Robust ANMF detection in non-centered impulsive background

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***Abstract*—One of the most general and acknowledged models for background statistics characterization is the family of elliptically symmetric distributions. They account for heterogeneity and non-Gaussianity of real data. Today, although non-Gaussian models are assumed for background modeling and design of detectors, the parameters estimation is usually performed using classical Gaussian-based estimators. This paper analyzes robust estimation techniques in a non-Gaussian environment and highlights their interest as an alternative to classical procedures for target detection purposes. The goal of this paper is to extend well-known detection methodologies to non-Gaussian framework, when the statistical mean is non-null and unknown. Furthermore, a theoretical closed-form expression for false-alarm regulation is derived and the Constant False Alarm Rate property is pursued to allow the detector to be independent of nuisance parameters. The experimental validation is conducted on simulations.**

***Index Terms*—Elliptical distributions, *M*-estimation, robustness, adaptive target detection, false alarm regulation.**

I. Introduction

MOST of the classical target detection methods are derived under Gaussian assumption (see for e.g. [1], [2], [3]) and need for the statistical characterization of the background usually through the first and second order parameters (i.e. the mean vector and the covariance matrix). However, in many applications, the outcome of the detection scheme to the background diverts from the theoretically expected Gaussian assumption. The actual distribution may have heavier tails compared to the expected distribution, and these tails will strongly increase the observed false-alarm rate of the detector. Introduced by Kelker in [4] and extended to the complex case in [5], the family of Elliptically Contoured distribution accounts for non-Gaussianity by providing a long tailed alternative to the multivariate Normal model. Although elliptical distributions have already been introduced for background modeling in wireless radio propagation problems [6], radar clutter echoes modeling [7], hyperspectral background characterization [8], [9], [10], the parameters estimation is often performed using classical Gaussian based estimators. For example, the covariance matrix is generally determined by the Sample Covariance Matrix (SCM) and the mean vector with the Sample Mean Vector (SMV): $\hat{\boldsymbol{\mu}}_{SMV} = \dfrac{1}{N}\sum_{i=1}^{N} \mathbf{x}_i$, $\hat{\boldsymbol{\Sigma}}_{SCM} = \dfrac{1}{N}\sum_{i=1}^{N}(\mathbf{x}_i - \hat{\boldsymbol{\mu}}_{SMV})(\mathbf{x}_i - \hat{\boldsymbol{\mu}}_{SMV})^H$. Indeed, these classical estimators correspond to the Maximum Likelihood Estimators (MLE) for Gaussian assumption. When the Gaussian hypothesis is not fulfilled, the performance of the detector will be deteriorated and the false-alarm rate will increase. Therefore, elliptical distributions can be used to derive robust estimators of the parameters and to evaluate the robustness of the statistical methods[11], [12]. Robust location and scatter *M*-estimators were firstly introduced as a generalization of the MLE. Up until now, they have been widely studied in statistics literature [13], [14], [15] and have been used in several signal processing applications, such as radar detection [16] and hyperspectral imaging [17]. When the underlying distribution is unknown, *M*-estimators provide an alternative approach for robust parameter estimation of elliptical populations. These can then be substituted in the detection scheme (two-step Generalized Likelihood Ratio Test (GLRT)) in place of the unknown mean vector and covariance matrix. This allows to obtain robust properties for target detection schemes derived under the Gaussian assumption. It is worth pointing out that, due to the occurrence of impulsive environments and outliers in real scenarios, robustness of statistical procedures is an essential design requirement for target detection. The detector's performance has been analyzed over simulations and real data in [18].

The main contributions presented in this work are the joint estimation of the covariance matrix and the mean vector of the data in robust estimation framework, and their associated Constant False Alarm Rate (CFAR) adaptive detection test. More precisely, a theoretical closed-form expression for false-alarm regulation is derived in Proposition III.1. We will show that the proposed detection method jointly used with robust estimates allow not only to overcome the heterogeneity and non-Gaussianity but also to reach the same performance than the conventional detector on homogeneous Gaussian background.

In the following, vectors (resp. matrices) are denoted by bold-faced lowercase letters (resp. uppercase letters). ^{H} represents the Hermitian operator (transpose conjugate). $\sim$ means "distributed as", $\overset{d}{=}$ stands for "shares the same distribution as", $\xrightarrow{d}$ denotes the convergence in distribution. $\mathbf{I}_m$ is the $m \times m$ identity matrix, j is the imaginary unit and $\Re(\mathbf{y})$ represents the real part of the complex vector $\mathbf{y}$. vec is the operator which transforms a $m \times n$ matrix into a vector of length mn, concatenating its n columns into a single column and Tr$(\cdot)$ denotes the trace operator.

II. Elliptical distributions

In this section, we present the class of complex elliptically contoured distributions [5], (see [19] for a complete review).

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Definition II.1. *A* m*-dimensional random complex vector* $\mathbf{z}$ *has a complex elliptical (CE) distribution if its characteristic function is of the form:*

$$\Phi_{\mathbf{z}}(\mathbf{c}) = \exp\left(j\,\Re(\mathbf{c}^H\,\boldsymbol{\mu})\right)\,\phi(\mathbf{c}^H\,\boldsymbol{\Sigma}\,\mathbf{c}) \tag{1}$$

for some function $\phi : \mathbb{R}^+ \to \mathbb{R}$*, called characteristic generator, a positive semidefinite matrix* $\boldsymbol{\Sigma}$*, called* scatter *matrix and* $\boldsymbol{\mu} \in \mathbb{C}^m$ *the* location *vector. We shall write* $\mathbf{z} \sim \mathcal{CE}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$*.*

From $\mathbf{z} \sim \mathcal{CE}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$, it does not follow that $\mathbf{z}$ has a probability density function (p.d.f) $f_{\mathbf{z}}(.)$. If it exists, then it has the form:

$$f_{\mathbf{z}}(\mathbf{z}) = |\boldsymbol{\Sigma}|^{-1} h_m\left((\mathbf{z}-\boldsymbol{\mu})^H\,\boldsymbol{\Sigma}^{-1}\,(\mathbf{z}-\boldsymbol{\mu})\right) \tag{2}$$

where h_m is any function such as (2) defines a p.d.f. in $\mathbb{C}^m$. The function h_m is usually called *density generator* and it is assumed to be only approximately known. In this case we may write $\mathcal{CE}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, h_m)$ instead of $\mathcal{CE}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$. The scatter matrix $\boldsymbol{\Sigma}$ describes the shape and orientation of the elliptical equidensity contours. If the second-order moment exists, then $\boldsymbol{\Sigma}$ reflects the structure of the covariance matrix $\mathbf{M}$, i.e. the covariance matrix is equal to the scatter matrix up to a scalar constant $\boldsymbol{\Sigma} = \gamma\,\mathbf{M}$. Note that while the scatter matrix is always defined up to a scalar constant, the covariance matrix does not exist for some CE distributions (e.g. Cauchy distribution). We are interested in the information contained in the structure of the matrix, but not on its scale. The class of elliptical distributions includes a large number of well-known distributions, as for instance the multivariate Gaussian [20], the *K*-distribution [21] or the multivariate *t*-distribution [5].

Let us now review some robust procedures particularly suited for estimating the scatter matrix and the mean vector of elliptical populations.

A. M-*Estimators*

When the density generator h_m is unknown, *M*-estimators provide a robust alternative for parameter estimation of elliptical populations. They have been introduced in this context as a generalization of MLE. Assume $\mathbf{z}_1, \mathbf{z}_2, ..., \mathbf{z}_N$ an independent and identically distributed (i.i.d.) sample from a $\mathcal{CE}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, h_m)$ with $N > m$. The complex *M*-estimators of location and scatter are defined as the joint solutions of:

$$\hat{\boldsymbol{\mu}}_N = \frac{\displaystyle\sum_{i=1}^{N} u_1(t_i)\,\mathbf{z}_i}{\displaystyle\sum_{i=1}^{N} u_1(t_i)}\,,\ \hat{\boldsymbol{\Sigma}}_N = \frac{1}{N}\sum_{i=1}^{N} u_2\left(t_i^2\right)\,(\mathbf{z}_i - \hat{\boldsymbol{\mu}}_N)\,(\mathbf{z}_i - \hat{\boldsymbol{\mu}}_N)^H, \tag{3}$$

where $t_i = \left((\mathbf{z}_i - \hat{\boldsymbol{\mu}})^H\,\hat{\boldsymbol{\Sigma}}^{-1}\,(\mathbf{z}_i - \hat{\boldsymbol{\mu}})\right)^{1/2}$ and $u_1(.), u_2(.)$ denote any real-valued *weighting functions* on the quadratic form t_i. Remark that t_i^2 here is nothing but the Mahalanobis distance and the main purpose of $u_1(.)$ and $u_2(.)$ is to attenuate the contribution of highly outlying samples. The choice of $u_1(.)$ and $u_2(.)$ does not need to be related to a particular elliptical distribution and therefore, *M*-estimators constitute a wide class of estimates that includes the MLE for the particular case $u_1(t) = -h_m'\left(t^2\right)/h_m\left(t^2\right)$ and $u_2\left(t^2\right) = u_1(t)$. Existence and uniqueness have been proven in the real case, provided functions $u_1(.), u_2(.)$ satisfy a set of general assumptions stated by Maronna [14]. Ollila has shown in [22] that these conditions hold also in the complex case. The solutions $(\hat{\boldsymbol{\mu}}_N, \hat{\boldsymbol{\Sigma}}_N)$ are estimates for the parameters $(\boldsymbol{\mu}, \boldsymbol{\Sigma}_0)$:

$$\boldsymbol{\Sigma}_0 = \mathbb{E}\left[u_2\left[(\mathbf{z}-\boldsymbol{\mu})^H\,\boldsymbol{\Sigma}_0^{-1}\,(\mathbf{z}-\boldsymbol{\mu})\right](\mathbf{z}-\boldsymbol{\mu})\,(\mathbf{z}-\boldsymbol{\mu})^H\right]. \tag{4}$$

For elliptical distributions, the implicit equation (4) admits a solution $\boldsymbol{\Sigma}_0$ and one has: $\boldsymbol{\Sigma} = \sigma\boldsymbol{\Sigma}_0$.
Hence, σ is obtained solving the following equation given in [23] and recapped here. Multiplying (4) by $\boldsymbol{\Sigma}_0^{-1}$ and taking trace yields:

$$\mathbb{E}\left[\psi_2(\sigma|\mathbf{t}|^2)\right] = m \tag{5}$$

where $\mathbf{t} \sim \mathcal{CE}(\mathbf{0}, \mathbf{I}_m)$ and $\psi_2(s) = s u_2(s)$.
We present now an example of M-estimators of location and scatter.

Fixed Point Estimators: The Fixed Point Estimators (FPE) firstly introduced in [24], satisfy the following implicit equations:

$$\hat{\boldsymbol{\mu}}_{FP} = \frac{\displaystyle\sum_{i=1}^{N} \frac{\mathbf{z}_i}{\left((\mathbf{z}_i - \hat{\boldsymbol{\mu}}_{FP})^H\,\hat{\mathbf{M}}_{FP}^{-1}\,(\mathbf{z}_i - \hat{\boldsymbol{\mu}}_{FP})\right)^{1/2}}}{\displaystyle\sum_{i=1}^{N} \frac{1}{\left((\mathbf{z}_i - \hat{\boldsymbol{\mu}}_{FP})^H\hat{\mathbf{M}}_{FP}^{-1}(\mathbf{z}_i - \hat{\boldsymbol{\mu}}_{FP})\right)^{1/2}}}, \tag{6}$$

$$\hat{\mathbf{M}}_{FP} = \frac{m}{N}\sum_{i=1}^{N} \frac{(\mathbf{z}_i - \hat{\boldsymbol{\mu}}_{FP})\,(\mathbf{z}_i - \hat{\boldsymbol{\mu}}_{FP})^H}{((\mathbf{z}_i - \hat{\boldsymbol{\mu}}_{FP})^H\,\hat{\mathbf{M}}_{FP}^{-1}\,(\mathbf{z}_i - \hat{\boldsymbol{\mu}}_{FP}))}, \tag{7}$$

which correspond to the particular cases of (3) for $u_1(t) = t^{-1}$ and $u_2\left(t^2\right) = m\,t^{-2}$. They are specified by fixed point equations and can be easily computed using a recursive algorithm. If the limit of the algorithm exists, it must be a solution. Although, the theoretical convergence of the procedure has not been proven, the empirical behavior is suitable.
The main results on the statistical properties of $\hat{\mathbf{M}}_{FP}$ are recalled here in the elliptical distribution framework (when $\boldsymbol{\mu}$ is assumed to be known): $\hat{\mathbf{M}}_{FP}$ is a consistent and unbiased estimate of $\boldsymbol{\Sigma}$; its asymptotic distribution is Gaussian and its covariance matrix is fully characterized in [25]; for N sufficiently large, $\hat{\mathbf{M}}_{FP}$ behaves as a Wishart matrix with $\frac{m}{m+1}N$ degrees of freedom (see [26] for a detailed performance analysis). Remark that the distribution of $\hat{\mathbf{M}}_{FP}$ does not depend on the true underlying elliptical distribution. In order to establish consistency and asymptotic normality, the population distribution cannot be too heavily concentrated around the center. Consistency and asymptotic distribution of $\hat{\mathbf{M}}_{FP}$ are demonstrated for the joint location-scatter estimation in the real case in [24]. For identification purposes, one may define a normalization constraint on the matrix estimate, e.g. $\mathrm{Tr}(\hat{\mathbf{M}}_{FP}) = m$.

B. *Asymptotic distribution of the* M*-estimators*

Let us specify the asymptotic distribution of the M-estimators. Assume $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_N$ an i.i.d. sample from a

$\mathcal{CE}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, h_m)$. Then, one has:

$$\sqrt{N}\text{vec}\big((\hat{\boldsymbol{\Sigma}}_N - \boldsymbol{\Sigma}_0)\big) \xrightarrow{d}$$
$$\mathcal{CN}\left(\mathbf{0}, \nu_1 \left(\boldsymbol{\Sigma}_0^T \otimes \boldsymbol{\Sigma}_0\right) + \nu_2 \,\text{vec}(\boldsymbol{\Sigma}_0)\text{vec}(\boldsymbol{\Sigma}_0)^H\right),$$

with:

$$\nu_1 = \frac{\mathbb{E}\left[\psi_2^2(\sigma t^2)\right]}{m\,(m+1)\left(1 + [m(m+1)]^{-1}\mathbb{E}\left[\sigma t^2 \psi_2'(\sigma t^2)\right]\right)^2}, \quad (8)$$

$$\nu_2 = \frac{\mathbb{E}\left[(\psi_2(\sigma t) - m\sigma)^2\right]}{\left(m + \mathbb{E}\left[\sigma t^2\, \psi_2'(\sigma t^2)\right]\right)^2} - \frac{\nu_1}{m}, \quad (9)$$

where $\nu_1 > 0$ and $\nu_2 \geq -\frac{\nu_1}{m}$ and σ solves Eq. (5).
Remark that the classical SCM verifies the previous conditions under Gaussian assumption taking $\nu_1 = 1$ and $\nu_2 = 0$. These results were investigated in [14], [23] firstly in the real case, and extended to the complex case in [19], [27] when the mean vector is completely known.

III. Main Results

In non-Gaussian context, the Adaptive Normalized Matched Filter (ANMF) detector, proposed in [28], takes advantage of its invariance properties and delivers better results than the other Gaussian-based detectors, such as the Adaptive Matched Filter (AMF) and the Kelly test, [29]. If the background does not fulfill the Gaussian hypothesis, the detector performance can be deteriorated, increasing the false-alarm rate. To account for heterogeneity and non-Gaussianity of the background, a possible way is to use of the ANMF test built with robust estimates. If some *a priori* knowledge of the noise statistics (e.g., K -distribution, t-distribution, etc.) is available, then $\boldsymbol{\Sigma}$ and $\boldsymbol{\mu}$ should be estimated by the MLE $\hat{\boldsymbol{\Sigma}}$ and $\hat{\boldsymbol{\mu}}$ of the assumed elliptical model. When there is no reliable statistical information on secondary data, they are assumed to be i.i.d. random samples from an unknown CE distribution. Then, practically any robust *M*-estimator could be used in the detection scheme. For heavy-tailed non-Gaussian background robustness of the selected *M*-estimator is perhaps the most important design criterion.

ANMF built with robust estimates

We replace the covariance matrix and the mean vector by robust M-estimators of scatter and location as they are consistent estimators of the covariance matrix up to a positive scalar and the mean vector within the class of CE distributions (two-step GLRT). Thus, the ANMF for both the mean vector and the covariance matrix estimation takes the form

$$\Lambda_{ANMF\,\hat{\boldsymbol{\Sigma}},\hat{\boldsymbol{\mu}}} = \frac{|\mathbf{p}^H\,\hat{\boldsymbol{\Sigma}}_N^{-1}\,(\mathbf{x} - \hat{\boldsymbol{\mu}}_N)|^2}{(\mathbf{p}^H\,\hat{\boldsymbol{\Sigma}}_N^{-1}\,\mathbf{p})\,\left((\mathbf{x} - \hat{\boldsymbol{\mu}}_N)^H\,\hat{\boldsymbol{\Sigma}}_N^{-1}\,(\mathbf{x} - \hat{\boldsymbol{\mu}}_N)\right)} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \lambda \quad (10)$$

where $\mathbf{p}$ is the steering vector describing the signal which is sought, $\hat{\boldsymbol{\mu}}_N$ and $\hat{\boldsymbol{\Sigma}}_N$ stand for any couple of M-estimators and N stresses their dependency with the number of secondary data. Note that the ANMF falls into the class of homogeneous functions $H(\cdot)$ of degree 0, i.e. the resulting detector does not depend on the scale factor of the matrix. When robust M-estimators are used jointly with the ANMF, the false-alarm can be regulated according to the following proposition.

Proposition III.1. *The theoretical relationship between the PFA and the threshold for the ANMF, built with M-estimators of location and scatter $\hat{\boldsymbol{\mu}}_N$ and $\hat{\boldsymbol{\Sigma}}_N$, is given by:*

$$PFA_{ANMF\,\hat{\boldsymbol{\Sigma}},\hat{\boldsymbol{\mu}}} = (1-\lambda)^{a-1}{}_2F_1\left(a, a-1; b-1; \lambda\right), \quad (11)$$

with $a = \sigma_1(N-1) - m + 2$ and $b = \sigma_1(N-1) + 2$, where N is the number of secondary data and m the dimension of the vectors. σ_1 is related to the particular choice of M-estimators and is obtained according to:

$$\sigma_1 = \frac{\mathbb{E}\left[\psi_2^2(\sigma t^2)\right]}{m\,(m+1)\left(1 + [m(m+1)]^{-1}\mathbb{E}\left[\sigma t^2 \psi_2'(\sigma t^2)\right]\right)^2}. \quad (12)$$

Proof: The "PFA-threshold" relationship for the ANMF detector is perfectly known in Gaussian context and when the used estimators are the SMV and a Wishart matrix obtained with the SCM. The "PFA-threshold" is derived in [30] and is recalled here:

$$PFA_{ANMF\,\hat{\boldsymbol{\Sigma}},\hat{\boldsymbol{\mu}}} = (1-\lambda)^{a-1}{}_2F_1\left(a, a-1; b-1; \lambda\right), \quad (13)$$

where $a = (N-1) - m + 2$ and $b = (N-1) + 2$ and ${}_2F_1(\cdot)$ is the hypergeometric function [31]. The statistical behavior of the M-estimators has been described in Section II-B. It has been shown that, for N large enough, M-estimators statistically behave as Wishart distributed matrices. Therefore, their distribution rely on the asymptotic variance of the considered M-estimators, σ_1, detailed above. Compared with the classical SCM-SMV, the only change is the correction factor σ_1 acting on $(N-1)$. For the general case of M-estimators, the relationship in Eq. (13) is verified for N large enough replacing $N-1$ by $(N-1)/\sigma_1$.
This allows to give an approximated "PFA-threshold" relationship for the M-estimators and for functions in the class of homogeneous functions of degree 0 (as it is the case for the ANMF). Indeed, we note that the test statistic in Eq. (10) stays the same if one substitutes $\hat{\boldsymbol{\Sigma}}$ by $\hat{\boldsymbol{\Sigma}}/\text{Tr}(\hat{\boldsymbol{\Sigma}})$. Thus, for N sufficiently large, the "PFA-threshold" relationship is given by Proposition III.1. This function only depends on the size m of the vectors and on the number N of secondary data used for the estimation stage as well as the asymptotic variance σ_1 of the considered M-estimators. ∎

Note that the variance of the mean estimator will not affect the distribution as it appears both at the numerator and the denominator and subsequently, it disappears.x

Although FPE do not belong to the class of M-estimators (as they do not satisfy the conditions of Maronna [14]), these results can also be extended to the FPE. The approximated "PFA-threshold" is obtained replacing in Eq. (13) $N-1$ by $\frac{m}{m+1}(N-1)$ as $\sigma_1 = \frac{m+1}{m}$ which is an extension of [32] for unknown mean vector.
The CFAR property of this detector in any heterogeneous and/or non-Gaussian background is reached when the FPE are used. On the other hand, as the background is non-Gaussian and/or heterogeneous, the statistical distribution of the ANMF

built with the SCM estimate cannot be predicted by Eq. (13), but it will surely vary with the background. The ANMF built with any M-estimators (and particularly FPE) does overcome the non-Gaussianity and/or heterogeneity of the data. This implies, thanks to the properties of the CE distributions, that the detector behaves according to the same distribution regardless of the true CE, i.e., it is distribution-free (see [33]). In addition, the asymptotic variance σ_1, which is always greater than 1, quantifies the loss of performance for the detector over optimality in Gaussian distributed background. Despite this small loss in Gaussian case, M-estimators bring robustness to the detection scheme and allow for false-alarm regulation within the class of CE distributions. The improvement pointed for false alarm regulation leads to a better performance in terms of probability of detection. Notably, the SNR required to detect a target can be considerably decreased.

IV. Simulations

In this section, we validate the theoretical analysis on simulated data. The experiments have been realized over a K-distribution with shape parameter $\nu = 0.5$ for $m = 10$ dimensional vectors, $N = 50$ secondary data and the computations have been made through 10^6 Monte-Carlo trials. The true covariance is chosen as a Toeplitz matrix whose entries are $\Sigma_{i,j} = \rho^{|i-j|}$ and where $\rho = 0.4$. The mean vector is arbitrarily set to have all entries equal to $(3 + 4j)$. Under a K-distribution, as shown on Fig. 1, the theoretical "PFA-threshold" relationship in Eq. (11) is in perfect agreement with the Monte-Carlo simulations for the FPE, while for the SCM-SMV, the theoretical "PFA-threshold" relationship in Eq. (13) is not valid anymore (since the Gaussian assumption is not respected anymore). We have left the theoretical "PFA-threshold" relationship for Gaussian estimators (black curve) for information.

It is worth pointing out that on both Gaussian and K-distribution contexts, the false alarm regulation for the FPE leads to the same results. Thus, the curve just depends on the size of the vector m and on the number of secondary data N. This fact emphasizes the maximal invariance obtained with the ANMF built with the FPE, i.e. the distribution of the detector under only background hypothesis remains the same for all different impulsive distributions within the class of CE distributions. This has been referred to as the CFAR property: one of the most attractive properties of the ANMF (FPE) detector is its distribution invariance to the true matrix (CFAR-matrix), to the true mean vector (CFAR-mean) and to the underlying distribution itself (CFAR texture), i.e. the distribution of the detector remains the same even for impulsive distributions and for different parameters of the corresponding distributions. This CFAR texture property is highlighted in Fig. 2. The experiments were conducted for $m = 3$ with $N = 21$ secondary data and the computations have been made through 10^6 Monte-Carlo trials for different impulsive distributions. Note that the detector behaves according to the same distribution regardless of the true elliptical distribution. However, this is the case only for the FPE and not for the other M-estimators.

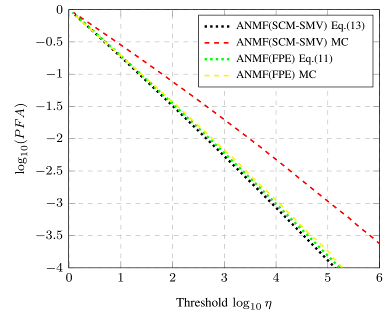

Fig. 1: PFA-threshold for the ANMF under a K-distribution with shape parameter $\nu = 0.3$ for $m = 10$ and $N = 50$ when (1) the SCM-SMV are used (red and black curves) (2) Proposition III.1: the FPE are used (yellow and green curves)

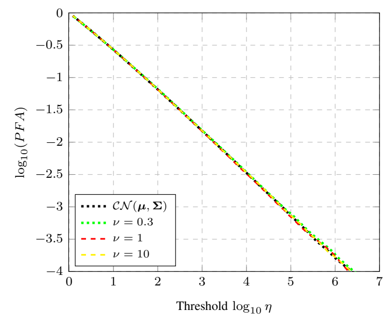

Fig. 2: PFA-threshold for the ANMF built with the FPE for different underlying distributions Gaussian and K-distribution with shape parameter ν and $m = 3$ and $N = 21$.

V. Conclusion

We have detailed the class of elliptically symmetric distributions as a general model for background characterization. Elliptical distributions account for heterogeneity and long tail distributions. Once established that real data can not fit a multivariate Normal distribution, the use of the Gaussian maximum likelihood estimates (SCM and SMV) do not provide the optimal parameter estimation. We propose the use of robust estimates for the mean vector and the covariance matrix. We have described the *M*-estimators, notably the FPE. The joint estimation of both parameters is a new challenging problem that opens many unknowns and it will be further investigated. We introduce here the use of these estimates on classical detection method. For false alarm regulation purposes, we have derived the theoretical relationship to set the proper threshold for a fixed probability of false alarm. Finally, we have validated the theoretical analysis over simulations. We conclude that the robust estimation tools presented in this paper offer a versatile alternative to Gaussian estimates. We remark that proposed *M*-estimators in Gaussian environment are capable of reaching the same results as the SCM and SMV. On the other hand, they outperform the classical estimation methods in case of non-Gaussian impulsive noise. This adaptability and their robustness make them suitable estimates in most scenarios and suggests its use in signal processing applications where covariance matrix and mean vector are both unknown and have to be estimated from the background.

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