Space varying PSF estimation: the telescope tomography approach

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Outlines

- **Telescope tomography**
- **Fast propagation through telescope**
- **Proximity operators for phase retrieval**

Stringent requirements on PSF knowledge

PSF shape biases the galaxy ellipticity measurement

- PSF varies across the wide field of view
- Field stars give a sample the PSF estimate
- PSF have to be estimated at galaxies position.

Two problems

- **estimation** of the PSF (using fields stars)
- **interpolation** of the PSF on galaxies positions

Telescope tomography

- **Space varying PSFs are totally described by aberrations of all mirrors**

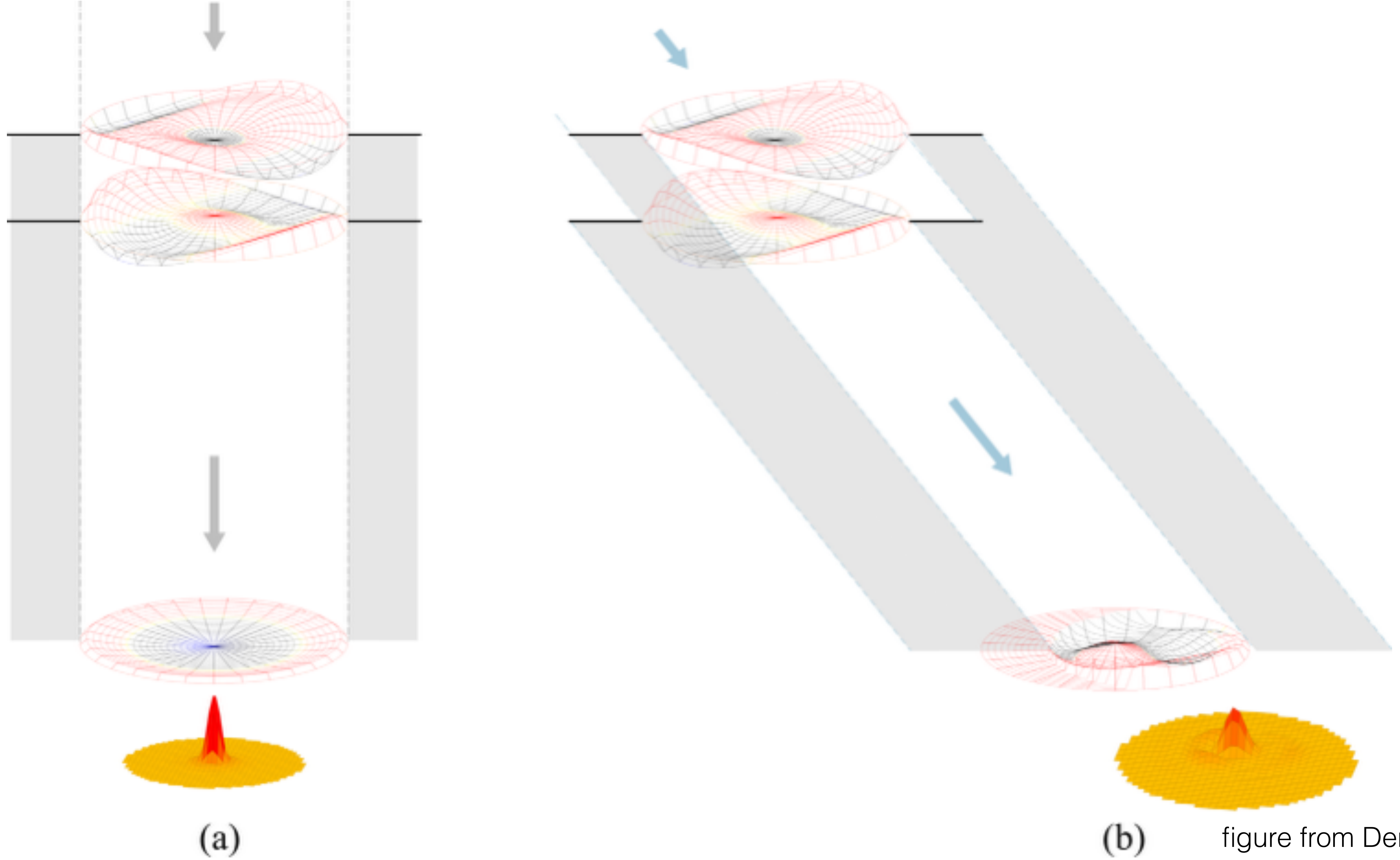


figure from Denis et al 2015

Mirror aberrations « sum up » in different manners across the field of view producing space varying PSFs.

Telescope tomography

Optical diffraction tomography

Reconstructing the 3D distribution of a biological sample using:

- illumination from different angles
- interferometric measurements of the wavefront in the detector plane
- light propagation modeled by BPM

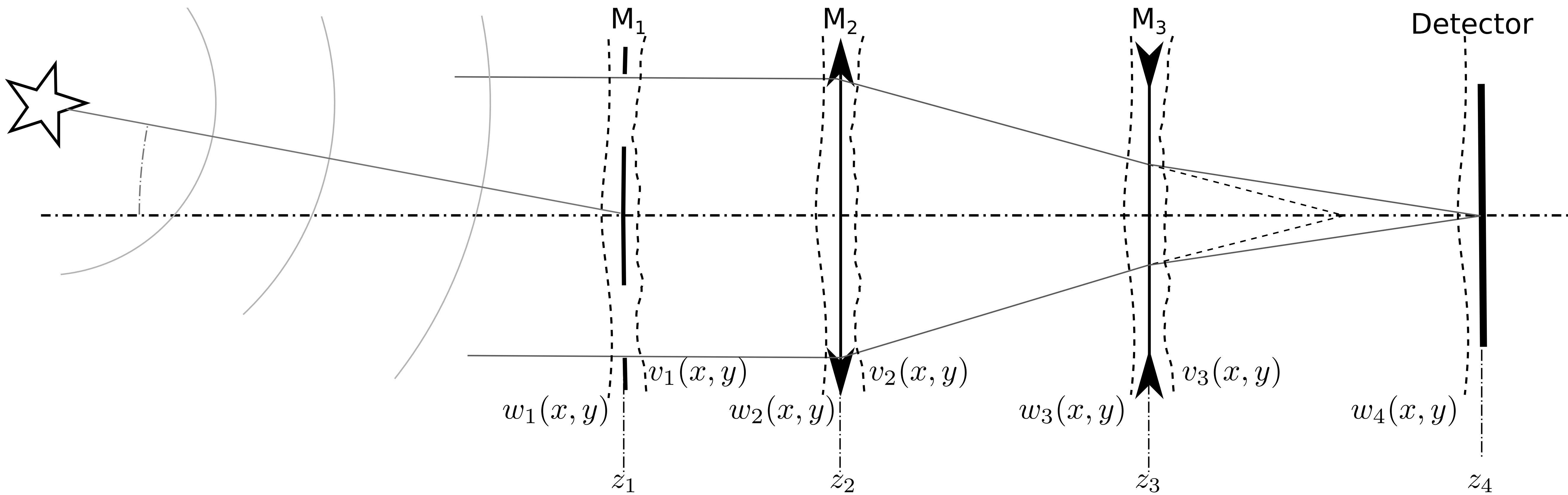
(Kamilov et al. *Optica* 2015)

Euclid telescope tomography

Estimating the aberrations of all telescope mirrors using:

- field stars: plane wave sources with different incoming angles
- star image (PSF): inline hologram of the wavefront
- light propagation modeled by BPM

Forward model



Wavefront $\boldsymbol{w}_k$ on optical surface k

$$\boldsymbol{w}_k(\boldsymbol{\alpha}) = \mathbf{H}_{k-1}\,\mathbf{M}_{k-1}\,\mathbf{A}_{k-1}(\boldsymbol{\alpha}_k)\,\boldsymbol{w}_{k-1}(\boldsymbol{\alpha})$$

- **Incoming wave**

  Star are are infinitely far: w_1 is a tilted plane wave

- **Propagation $\mathbf{H}_k$**

Forward model

Wavefront $\boldsymbol{w}_k$ on optical surface k

$$\boldsymbol{w}_k(\boldsymbol{\alpha}) = \mathbf{H}_{k-1}\,\mathbf{M}_{k-1}\,\mathbf{A}_{k-1}(\boldsymbol{\alpha}_k)\,\boldsymbol{w}_{k-1}(\boldsymbol{\alpha})$$

- **Mirrors** M_k

  — change optical path
  — cuts light outside of the pupil

- **Aberrations** $\mathbf{A}_k$

Introduced by polishing errors and misalignment:
Expressed on a suitable basis: $\mathbf{A}_k(\boldsymbol{\alpha}_k) = \mathrm{diag}\left(\exp\left(\imath\,\mathbf{Z}_k\,\boldsymbol{\alpha}_k\right)\right)$.

- **Measurements**

$$d_n = \left|w_{K,n}\right|^2 + e_n$$

Aberrations estimation

Wavefront $\boldsymbol{w}_k$ on optical surface k

$$\boldsymbol{w}_k(\boldsymbol{\alpha}) = \mathbf{H}_{k-1}\,\mathbf{M}_{k-1}\,\mathbf{A}_{k-1}(\boldsymbol{\alpha}_k)\,\boldsymbol{w}_{k-1}(\boldsymbol{\alpha})$$

$$d_n = |w_{K,n}|^2 + e_n$$

- **‘Three times’ non-linear problem**
  - — Aberrations expressed in phase
  - — Propagation through multiple mirrors
  - — Intensity only measurements

- **Huge size problem**
  - — 24000 x 24000 pixels
  - — PSF undersampled by a factor 2
  - — at least 1800 stars per images

Framework

- **The reconstruction problem**

$$\boldsymbol{\alpha}^{+} = \arg\min_{\boldsymbol{\alpha}} \sum_{s=1}^{S} \sum_{n=1}^{N} \frac{1}{\sigma_n^2} \left(\left| w_{K,n}^{s}(\boldsymbol{\alpha}) \right|^2 - d_n^s \right)^2$$

- **The constrained formulation**

$$\boldsymbol{\alpha}^{+} = \arg\min_{\boldsymbol{\alpha}} \sum_{s=1}^{S} \sum_{n=1}^{N} \frac{1}{\sigma_n^2} \left(|t_n^s|^2 - d_n^s \right)^2 \quad \text{subject to } \boldsymbol{w}_K^s(\boldsymbol{\alpha}) = \boldsymbol{t}^s$$

- **Its augmented Lagrangian form**

$$\mathcal{L}(\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{u}) = \sum_{s=1}^{S} \sum_{n=1}^{N} \frac{1}{\sigma_n^2} \left(|t_n^s|^2 - d_n^s \right)^2 + \frac{\rho}{2} \sum_{s=1}^{S} \|\boldsymbol{w}_K^s(\boldsymbol{\alpha}) - \boldsymbol{t}^s - \boldsymbol{u}^s\|_2^2$$

Augmented Lagrangian without alternating direction

A hierarchical optimization

$$\boldsymbol{\alpha} = \arg\min_{\boldsymbol{\alpha}} \sum_{s=1}^{S} \sum_{n=1}^{N} \left| w^s_{K,n}(\boldsymbol{\alpha}) - t^s_n(\boldsymbol{\alpha}) - u^s_n \right|^2$$

$$\text{with } t^s_n(\boldsymbol{\alpha}) = \arg\min_{t \in \mathbb{C}} \frac{1}{\sigma^2_n} \left(|t|^2 - d^s_n \right)^2 + \frac{\rho}{2} \left| t - w^s_{K,n}(\boldsymbol{\alpha}) + u^s_n \right|^2$$

Solved using a continuous iterative optimization method (e.g. VMLMB).

(Mourya et al. *ICIP* 2015)

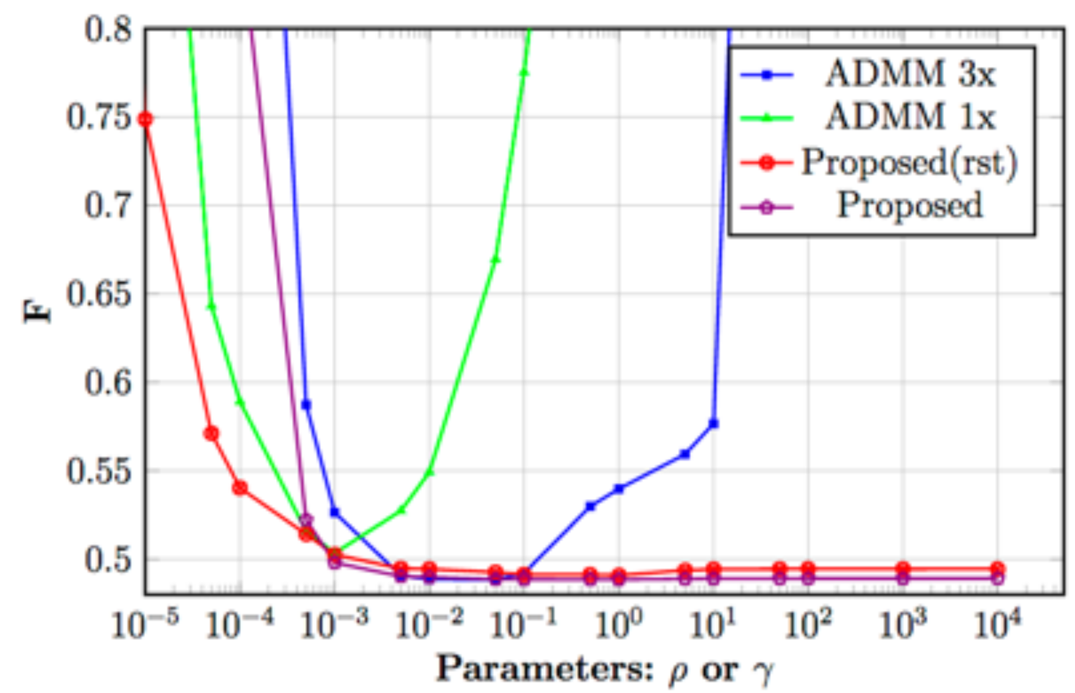
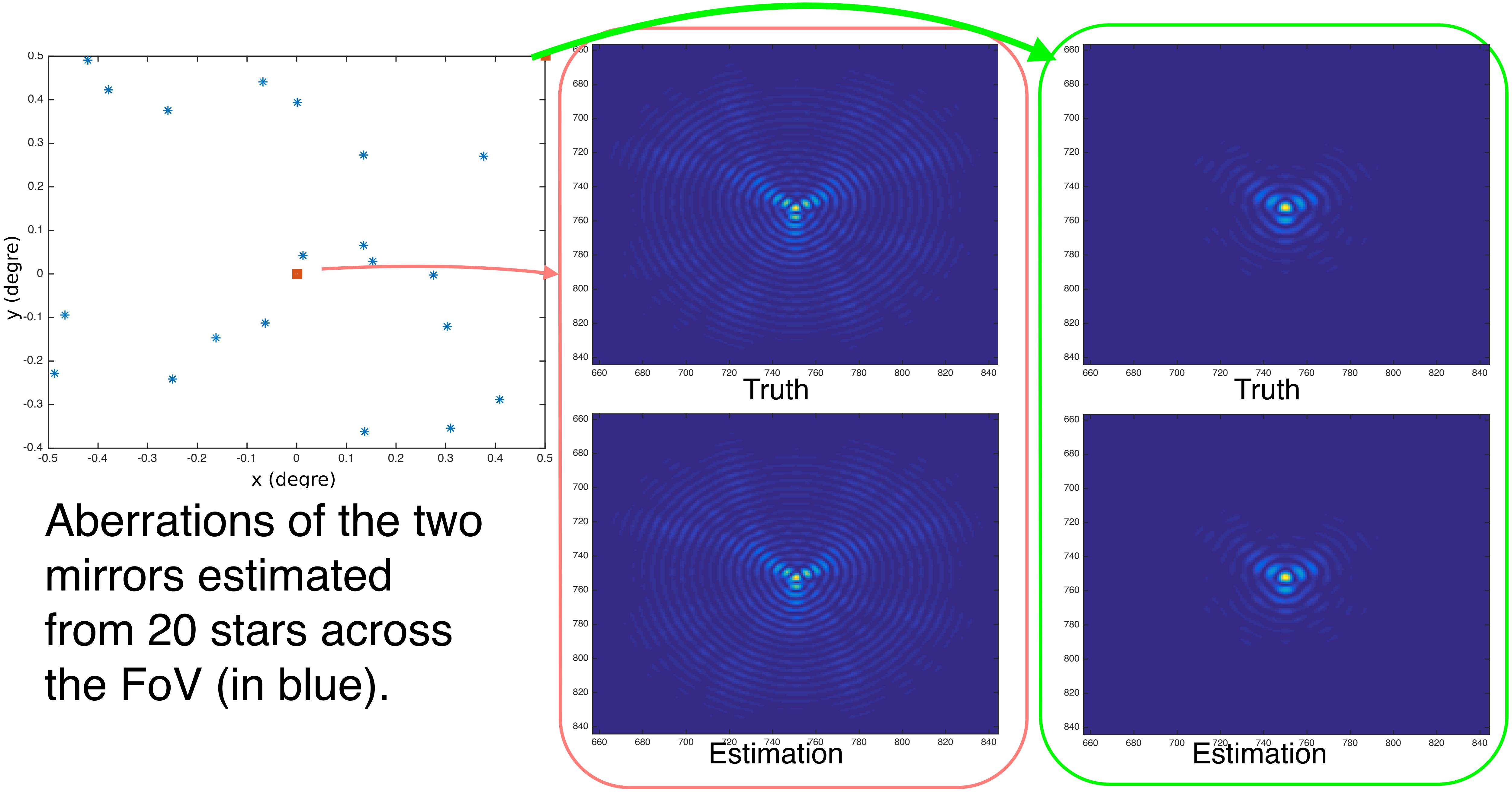


Fig. 2: Influence of the penalty parameters on the convergence speed: function cost at 1500 FFT evaluations vs. penalty parameters, ρ for **ADMM 3x** (ν and γ at possible best values), and γ for remaining two methods.

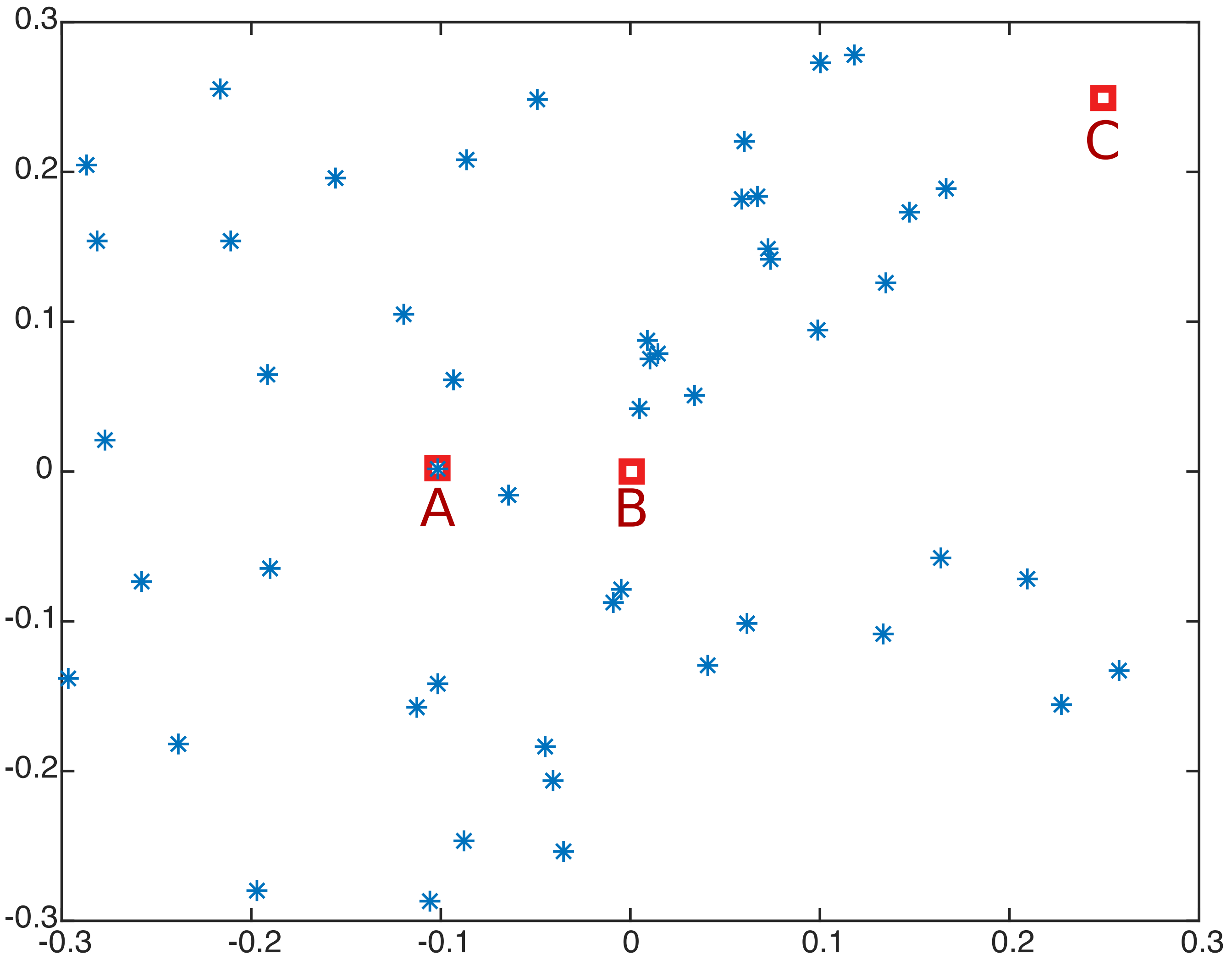
Preliminary results on HST-like telescope

strong aberrations (narrow band, no noise)

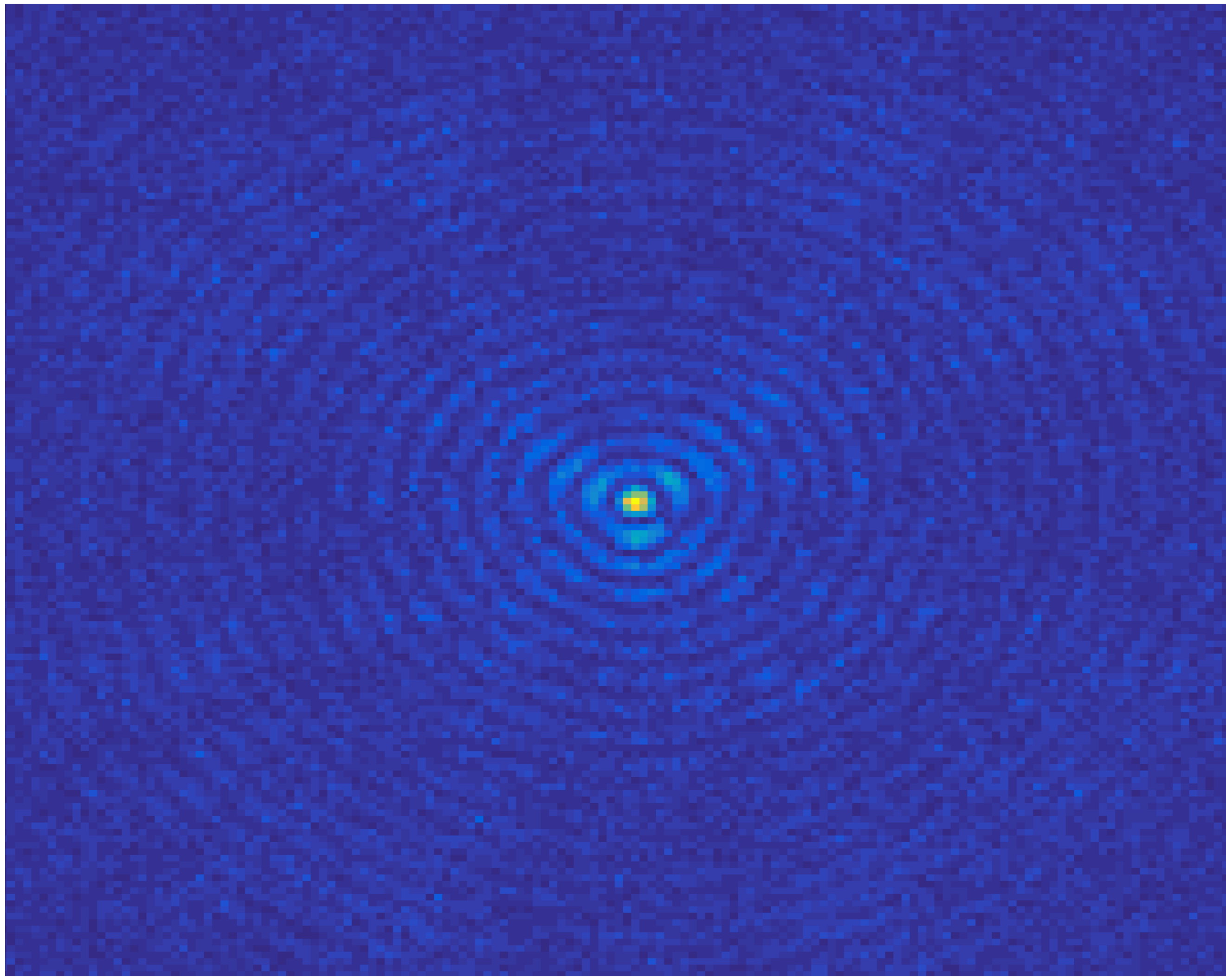


Aberrations of the two mirrors estimated from 20 stars across the FoV (in blue).

Results on HST-like telescope with noise



Field of view



Star A

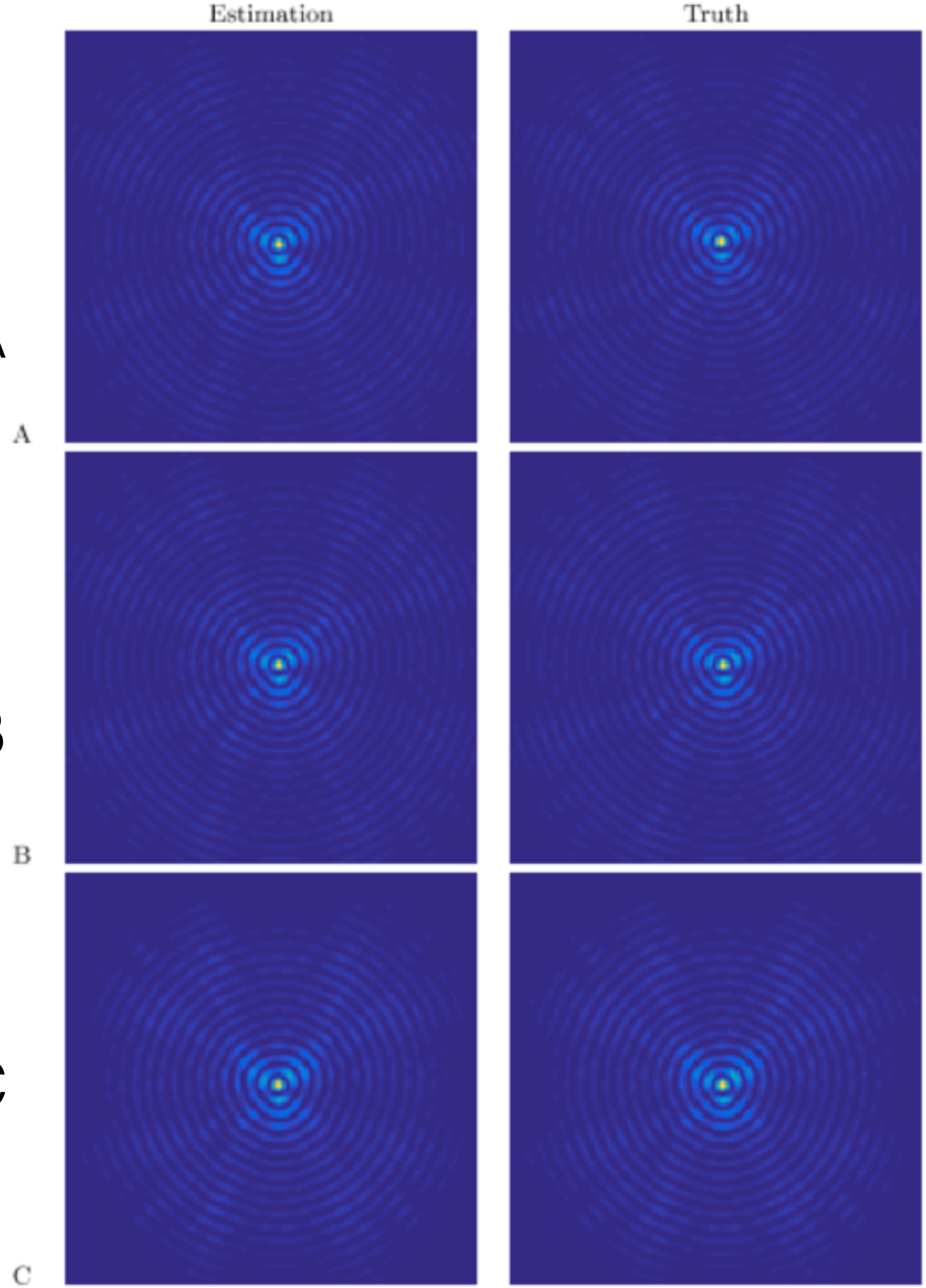
50 stars, 26 000 photons per star (256 at max), 5 e- of noise

Strong aberrations on both mirrors

A

B

C



Fast light propagation through telescope

$$\boldsymbol{\alpha} = \arg\min_{\boldsymbol{\alpha}} \sum_{s=1}^{S} \sum_{n=1}^{N} \left| w_{K,n}^{s}(\boldsymbol{\alpha}) - t_{n}^{s}(\boldsymbol{\alpha}) - u_{n}^{s} \right|^{2}$$

$$\text{with } t_{n}^{s}(\boldsymbol{\alpha}) = \arg\min_{t \in \mathbb{C}} \frac{1}{\sigma_{n}^{2}} \left(|t|^{2} - d_{n}^{s} \right)^{2} + \frac{\rho}{2} \left| t - w_{K,n}^{s}(\boldsymbol{\alpha}) + u_{n}^{s} \right|^{2}$$

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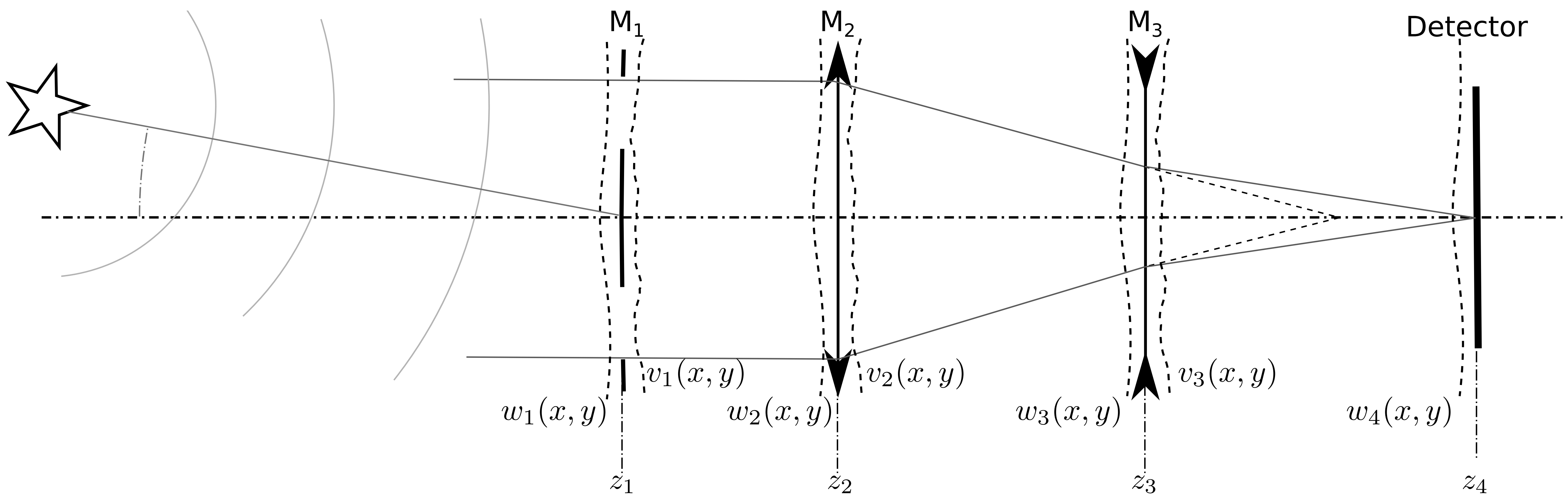
Forward model



Wavefront $\boldsymbol{w}_k$ on optical surface k

$$\boldsymbol{w}_k(\boldsymbol{\alpha}) = \mathbf{H}_{k-1}\,\mathbf{M}_{k-1}\,\mathbf{A}_{k-1}(\boldsymbol{\alpha}_k)\,\boldsymbol{w}_{k-1}(\boldsymbol{\alpha})$$

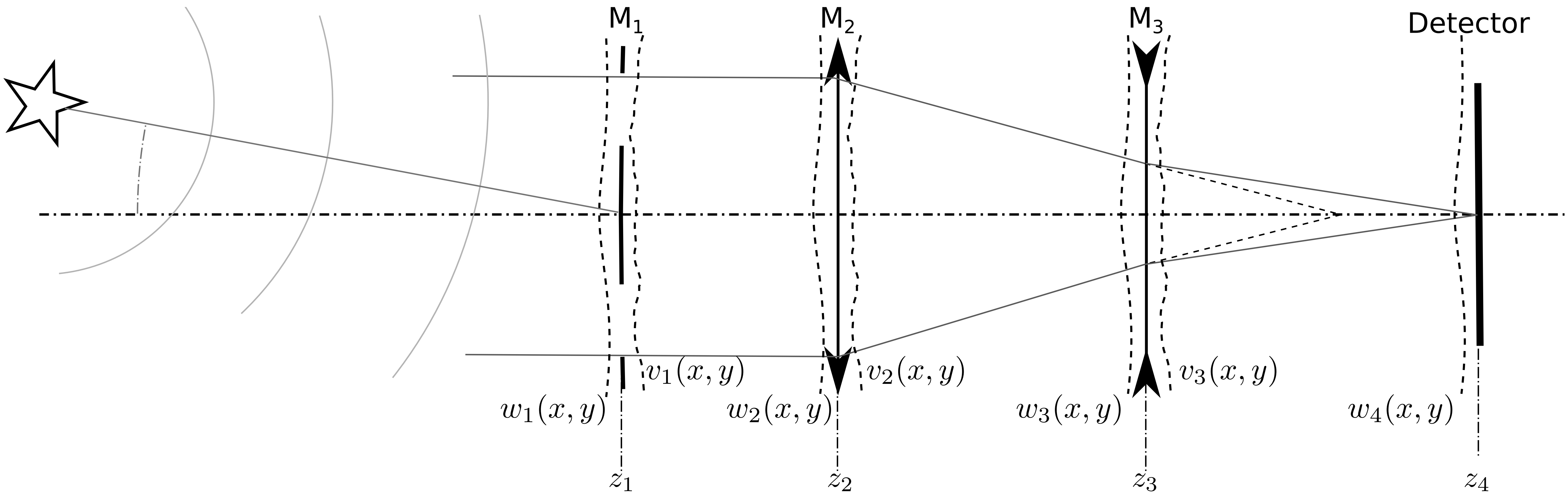
- **Incoming wave**

  Star are are infinitely far: w_1 is a tilted plane wave

  $$w_1(x,y) = \exp\left(\imath\left(x\,\sin(\theta_1)/\lambda + y\,\sin(\theta_2)/\lambda\right)\right)$$

  a **shift** in Fourier domain

Forward model



$$\boldsymbol{w}_k(\boldsymbol{\alpha}) = \mathbf{H}_{k-1}\,\mathbf{M}_{k-1}\,\mathbf{A}_{k-1}(\boldsymbol{\alpha}_k)\,\boldsymbol{w}_{k-1}(\boldsymbol{\alpha})$$

- **Propagation: paraxial / Fresnel approximation**

$$\mathbf{H}_k = \mathbf{F}^{-1} \cdot \mathrm{diag}\left(\hat{\boldsymbol{h}}_{(z_{k+1}-z_k)}\right) \cdot \mathbf{F}$$

$$\hat{h}(\boldsymbol{\omega}, z) = \mathrm{e}^{-\jmath\, z\, \frac{\|\boldsymbol{\omega}\|^2}{2\,k}}$$

Mirrors Model

Mirror transmittance

$$\mathbf{M}_k = \mathrm{diag}(m_k)$$

$$m_k(\boldsymbol{x}) = \begin{cases} \exp\left(\imath\, 2\, k \dfrac{\|\boldsymbol{x}\|^2}{R_k + \sqrt{R_k^2 - (1-\epsilon_k^2)\, \|\boldsymbol{x}\|^2}}\right), & \text{if } \|\boldsymbol{x}\| \leq D_k \\ 0\,, & \text{otherwise.} \end{cases}$$

diameter

curvature

eccentricity

Aberrations

$$\mathbf{A}_k = \mathrm{diag}\left(a_k(\boldsymbol{\alpha}_k)\right)$$

$$a_k(\boldsymbol{\alpha}_k) = \exp\left(\imath\, 2\, \mathbf{Z}_k^T \cdot \boldsymbol{\alpha}_k\right)$$

aberration coefficients

suitable basis (e.g. Zernike)

Sampling issues

- **Mirror transmittance**

Maximum instant frequency for mirror k

$$\nu_k^{\max} = \frac{D_k}{\lambda \sqrt{D_k^2 \, (\epsilon_k^2 - 1) + 4 R_k^2}} \, ,$$

- **Propagator**

Sampling requirements

Maximum instant frequency for propagator k

$L_k^{\max} = z \, \rho \, \lambda$ = width of the field of view needed

Periodicity issues

$$L_i = D_i + \frac{(z_{i+1} - z_i) \, \lambda}{2 \, \pi^2 \, D_i \, \vartheta}$$

fraction of energy that spills from an aperture to another

Sampling issues

Numbers...

For HST (M1 to M2):

- maximum pixel size: $2.3\,\mu\mathrm{m}$
- number of pixels $>> 10^6 \times 10^6$

Solution

inspired from Sziklas & Siegman, 1975

Coordinate transform: propagation in a prime domain:

$$\boldsymbol{w}_k(\boldsymbol{x}) = \boldsymbol{w}'_k(\boldsymbol{x}')\exp\left(\imath\left(\beta_k\left\|\boldsymbol{x}\right\|^2 + <\boldsymbol{\theta}'_k.\boldsymbol{x}>\right)\right)$$

Mirrors are much smoother in primed coordinate

maximum pixel size: 3.8 mm -> 1500 x 1500 pixels

Proximity operator for phase retrieval

$$\boldsymbol{\alpha} = \arg\min_{\boldsymbol{\alpha}} \sum_{s=1}^{S} \sum_{n=1}^{N} \left| w_{K,n}^{s}(\boldsymbol{\alpha}) - t_{n}^{s}(\boldsymbol{\alpha}) - u_{n}^{s} \right|^{2}$$

h $$t_{n}^{s}(\boldsymbol{\alpha}) = \arg\min_{t \in \mathbb{C}} \frac{1}{\sigma_{n}^{2}} \left(|t|^{2} - d_{n}^{s} \right)^{2} + \frac{\rho}{2} \left| t - w_{K,n}^{s}(\boldsymbol{\alpha}) + u_{n}^{s} \right|^{2}$$

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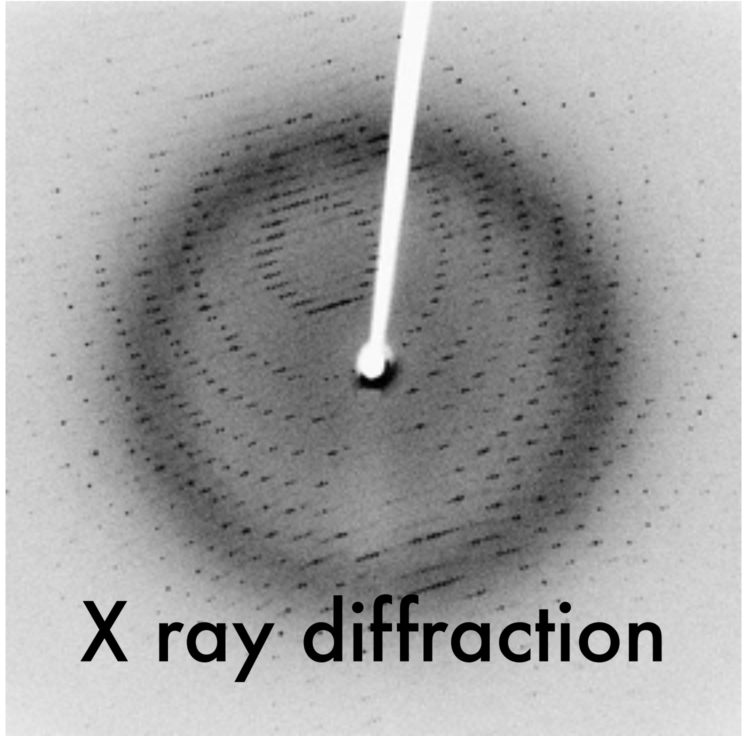
Phase retrieval in coherent imaging

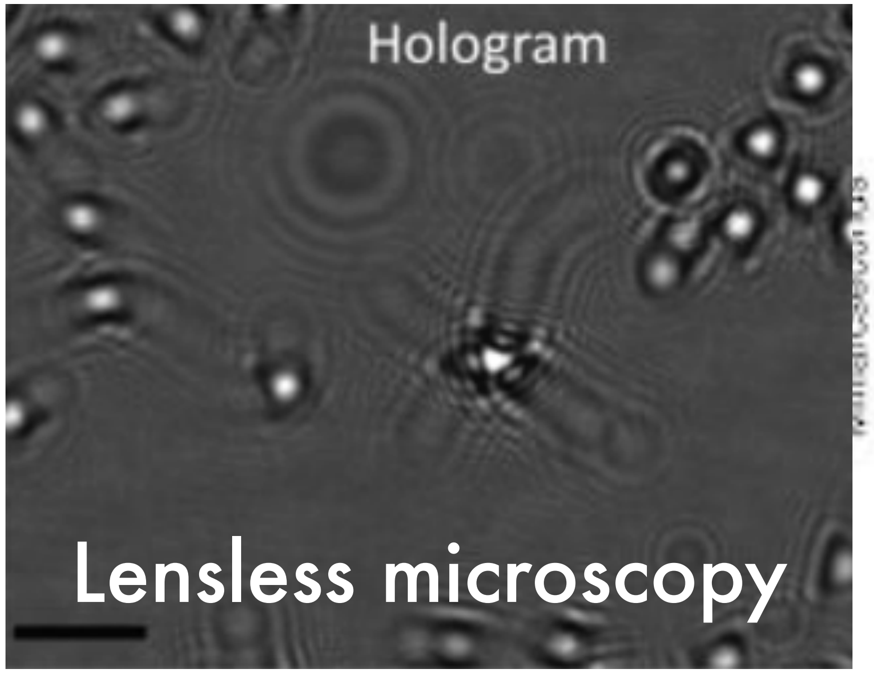
■ A twofold forward model

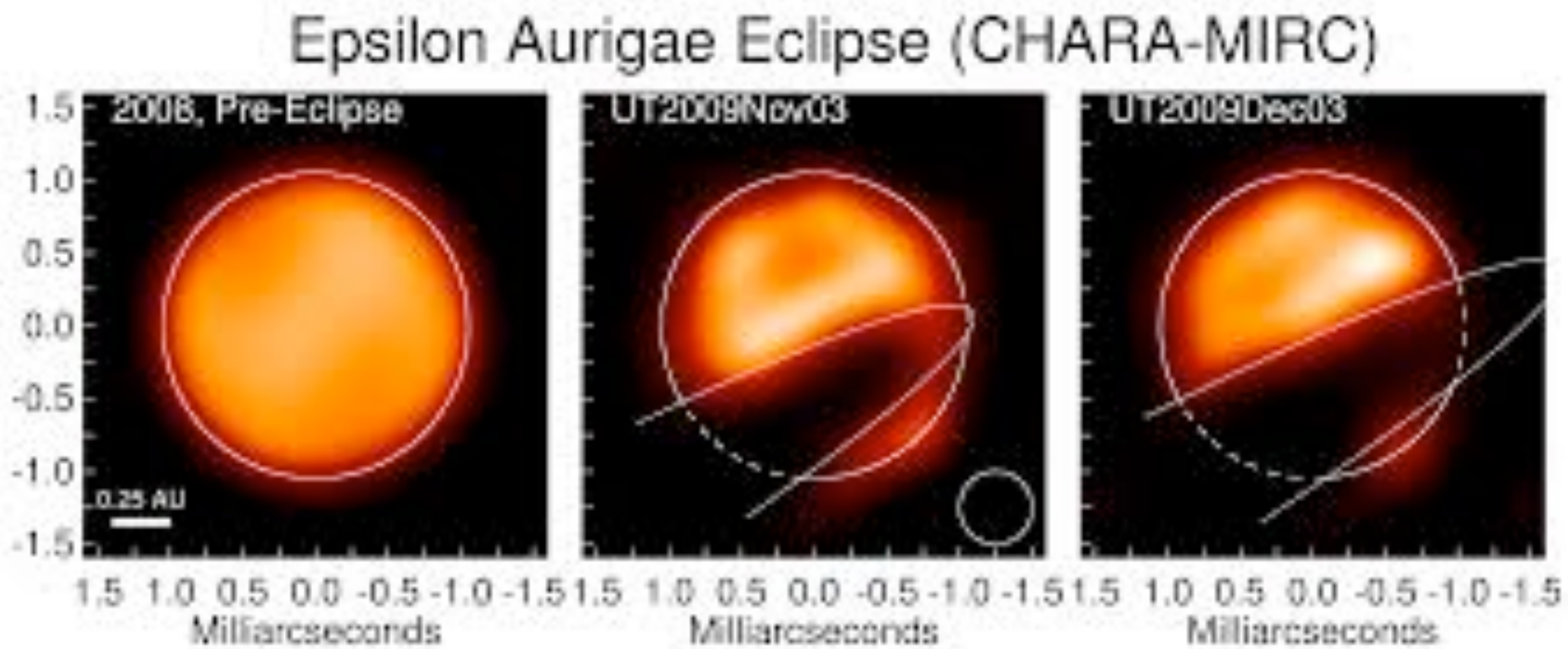
Wave optics model $\boldsymbol{y} = \mathcal{H}(\boldsymbol{x})$

Intensity measurements $d_k = |y_k|^2 + n_k$



■ Arise in numerous applications





Stellar interferometry

The phase retrieval problem

- **Estimating complex valued signal from intensity**

$$d_2 = |x_2|^2 + n$$

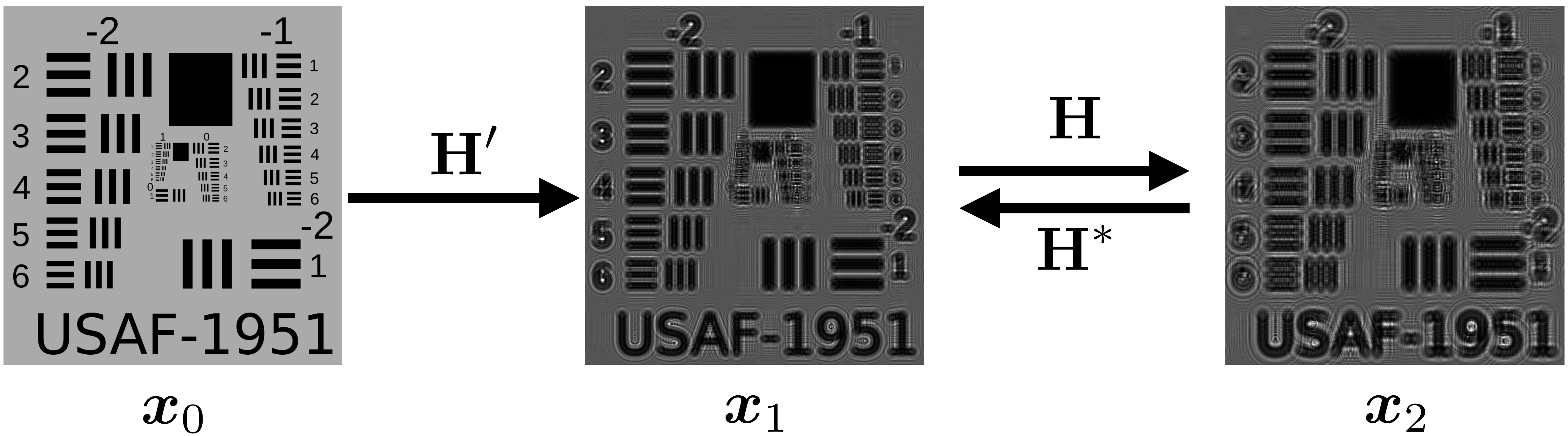
- **An abundant literature**

- iterative projections [Misell 1973, Fienup 1980, Bauschke 2003,…]
- semi-definite programming (Phase lifting) [Candes 2013, Fogel 2013]
- phase marginalization in variational bayesian approach [Dremeau 2015]

Gerchberg Saxton Algorithm

The simplest setup

- estimate the wavefront $\mathbf{x_1}$ in plane z_1 given intensity measurements at two different depth z_1 and z_2



Propagation

- Propagation using either Fresnel or Fraunhoffer approximation:

$$\boldsymbol{x}_2 = \mathbf{H}\boldsymbol{x}_1$$

$$\boldsymbol{x}_1 = \mathbf{H}^*\boldsymbol{x}_2$$

Gerchberg Saxton Algorithm

- **A non convex POCS algorithm**

Algorithm 1. Gerchberg-Saxton algorithm

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procedure GS($\boldsymbol{d}_A$, $\boldsymbol{d}_B$)

  $\boldsymbol{x}^{(0)} = \sqrt{\boldsymbol{d}_A}$ ▷ Initialization

  for $n = 1, 2, \ldots,$ maxiter **do**

    $\boldsymbol{y}^{(n+1/2)} = \mathbf{H} \cdot \boldsymbol{x}^{(n)}$ ▷ Propagation to the z_B plane

    $\boldsymbol{y}^{(n)} = P_B(\boldsymbol{y}^{(n+1/2)})$ ▷ Projection

    $\boldsymbol{x}^{(n+1/2)} = \mathbf{H}^{-1} \cdot \boldsymbol{y}^{(n)}$ ▷ Back propagation to the z_A plane

    $\boldsymbol{x}^{(n)} = P_A(\boldsymbol{x}^{(n+1/2)})$ ▷ Projection

  return $\boldsymbol{x}^{(\text{maxiter})}$ ▷ The complex amplitude in the z_A plane

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- **Projection step**

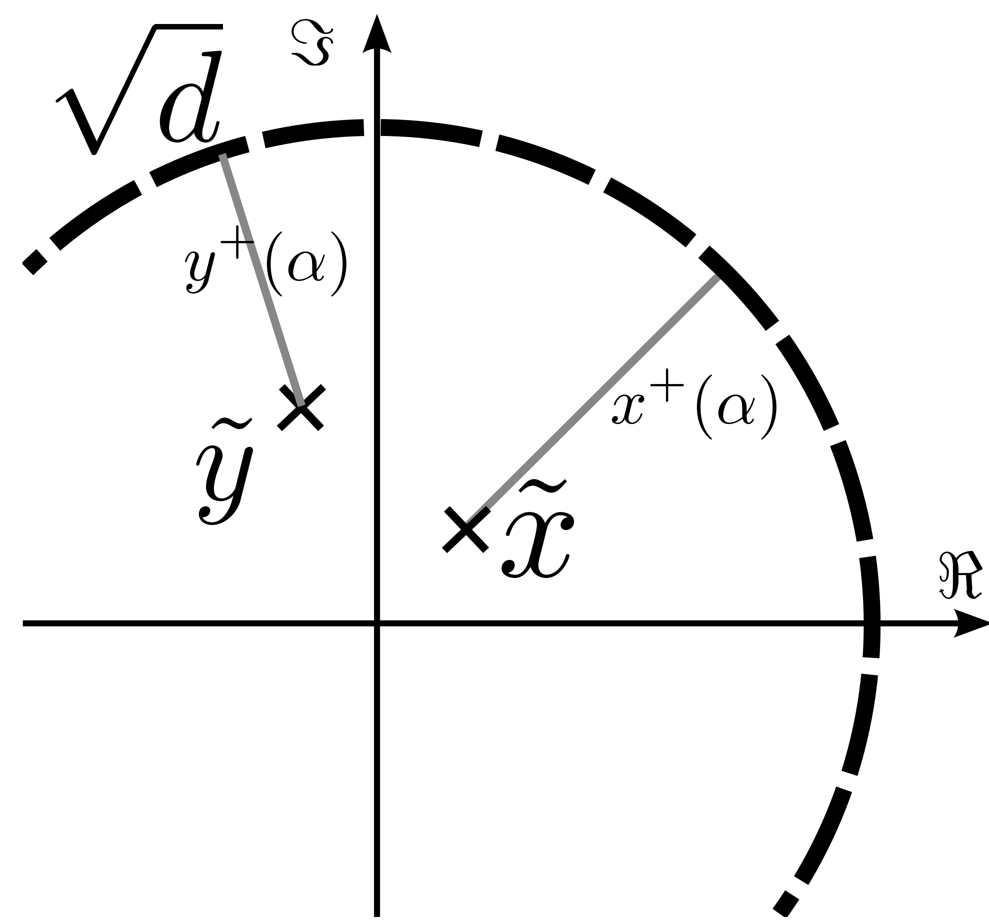
$$P(x_k \,|d_k) = \begin{cases} \frac{x_k}{|x_k|}\sqrt{d_k}\,, & \text{if } |x_k| > 0 \\ \sqrt{d_k}\,, & \text{otherwise.} \end{cases}$$

Modulus only proximity operator

$$\mathrm{prox}_\ell(\widetilde{\rho}) = \operatorname*{arg\,min}_{\rho \geq 0} \left\{ \ell(d - \rho^2) + \frac{1}{2}(\rho - \widetilde{\rho})^2 \right\}$$

Properties

- lower semi-continuous
- prox-bounded
- not prox-regular in **{0}**
- non-expansive only where $\widetilde{\rho} > \sqrt{d}$



Proximity operator for intensity

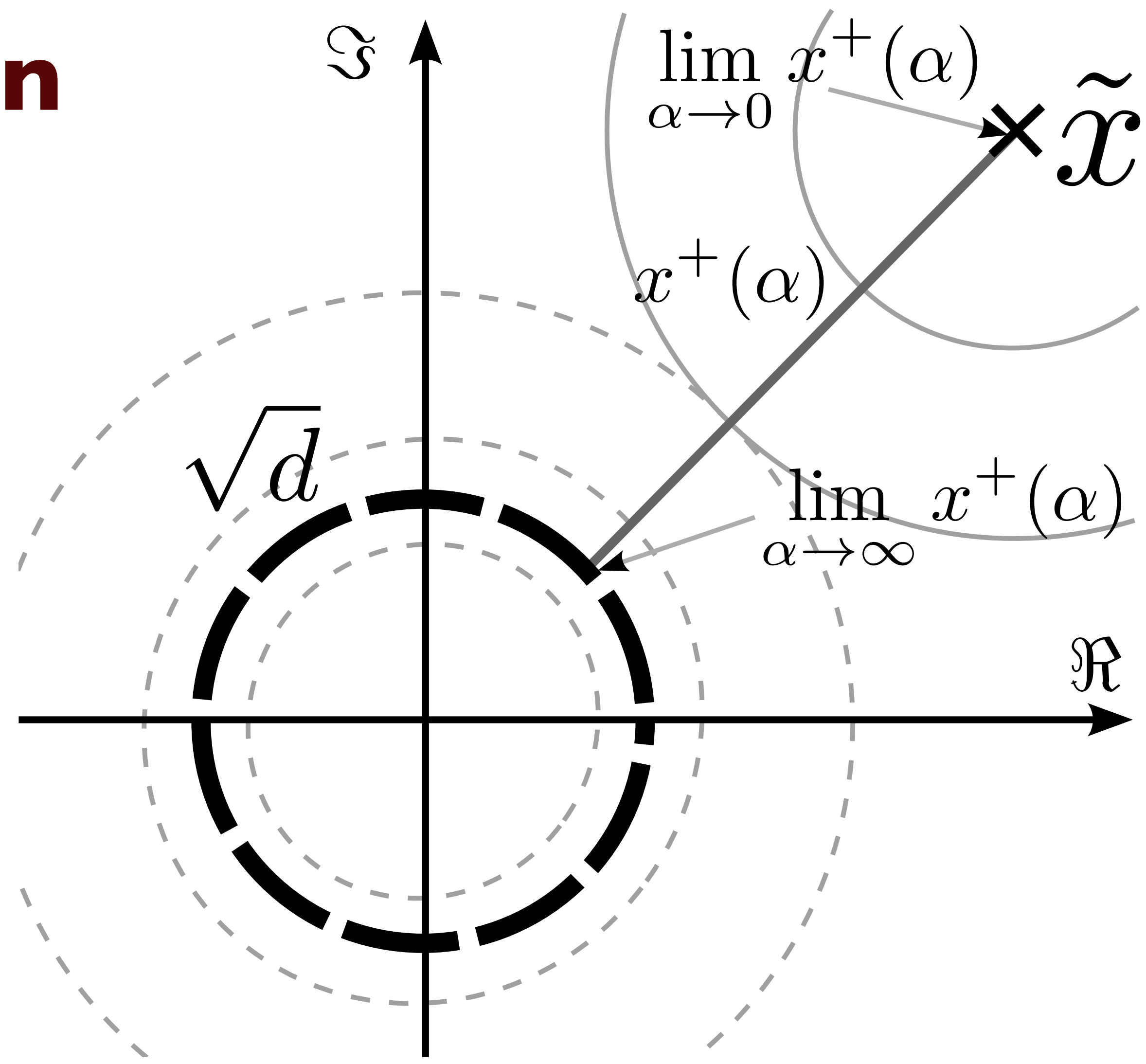
- **Maximum likelihood** $f_k(x) = \ell_k(d_k - |x_k|^2)$
  - is the neg-loglikelihood of the noise distribution

- **Phase independent solution**

$$\rho^+ \mathrm{e}^{\mathrm{i}\phi^+} = \mathrm{prox}_{\alpha\, f}\left(\rho\, \mathrm{e}^{\mathrm{i}\phi}\right)$$

$$\phi^+ = \phi$$

$$\rho^+ = \operatorname*{arg\,min}_{\rho \geq 0} \left(\alpha f(\rho) + \frac{1}{2}(\rho - \widetilde{\rho})^2\right)$$



Proximity operator for intensity

- **Gaussian likelihood** $f(x) = \frac{1}{\sigma^2}\left(|x|^2 - d\right)^2$

$$\operatorname{prox}(\widetilde{\rho}) = \arg\min_{\rho \geq 0} \left\{ \frac{1}{\sigma^2}(\rho^2 - d)^2 + \frac{1}{2}(\rho - \widetilde{\rho})^2 \right\}$$

$$\begin{aligned} q_G(\rho) &= \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\sigma^2}(\rho^2 - d)^2 + \frac{1}{2}(\rho - \widetilde{\rho})^2\right) \\ &= 4\,\frac{1}{\sigma^2}\,\rho^3 + \rho\left(1 - 4\,\frac{1}{\sigma^2}\,d\right) - \widetilde{\rho}\,. \end{aligned}$$

3rd order polynomial with only one root in R^+ computed using Cardano Method

Proximity operator for intensity

- **Poisson likelihood** $f(x) = |x|^2 - d\,\log\left(|x|^2 + b\right)$

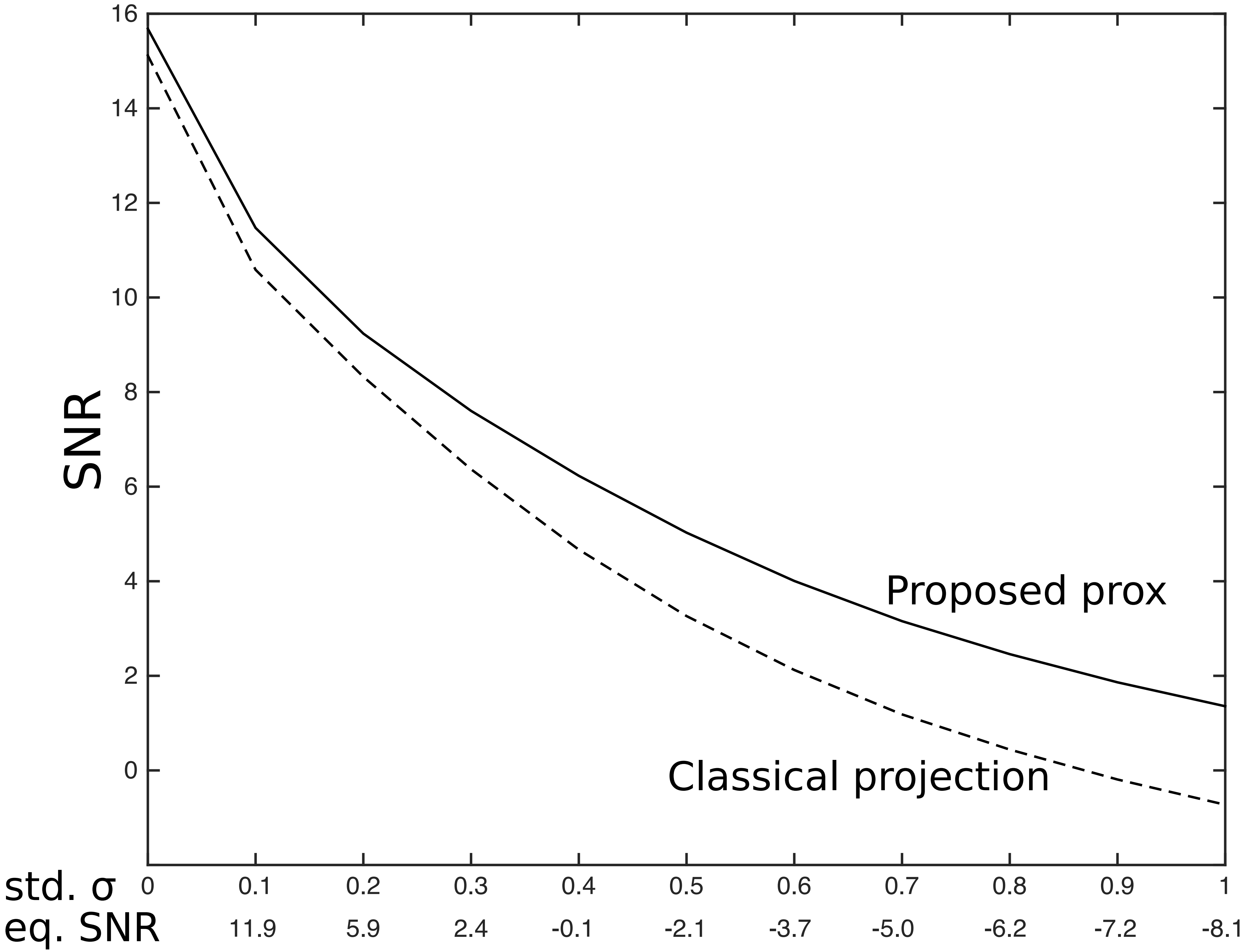
$$q_P(\rho) = \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\alpha\,f(\rho) + \frac{1}{2}\,(\rho - \widetilde{\rho})^2\right)$$
$$= (2\,\alpha + 1)\,\rho^3 - \widetilde{\rho}\,\rho^2 + ((2\,\alpha + 1)\,b - 2\,\alpha\,d)\;\rho - b\,\widetilde{\rho}$$

3rd order polynomial with only one root in R^+
computed using Cardano Method

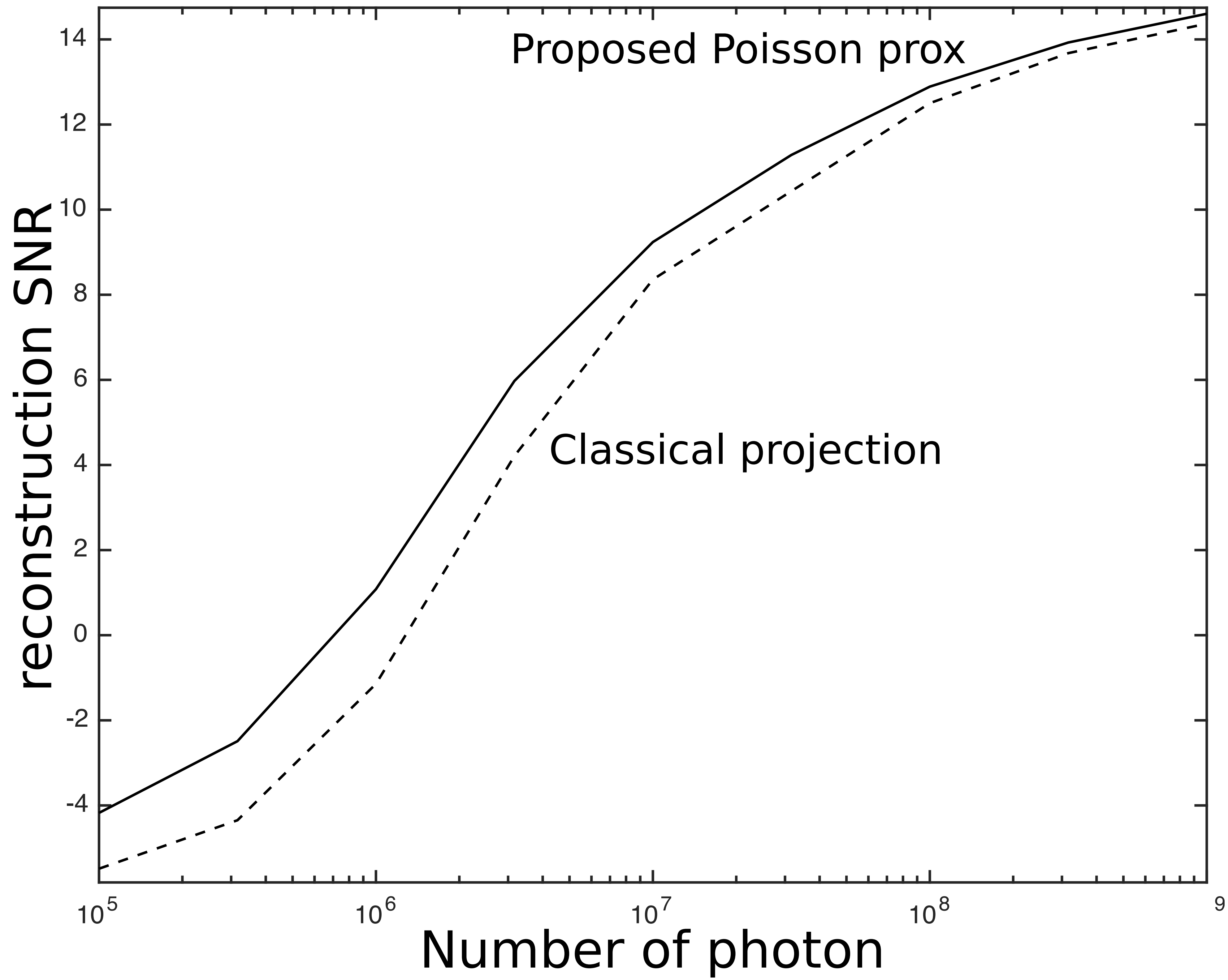
if b=0 then it reduces to a 2nd order polynomial

$$\rho^+ = \frac{\widetilde{\rho} + \sqrt{8\,d\,\alpha\,(1 + 2\,\alpha) + \widetilde{\rho}^2}}{2 + 4\,\alpha}\,.$$

Numerical results : Gaussian noise



Numerical results : Poisson noise



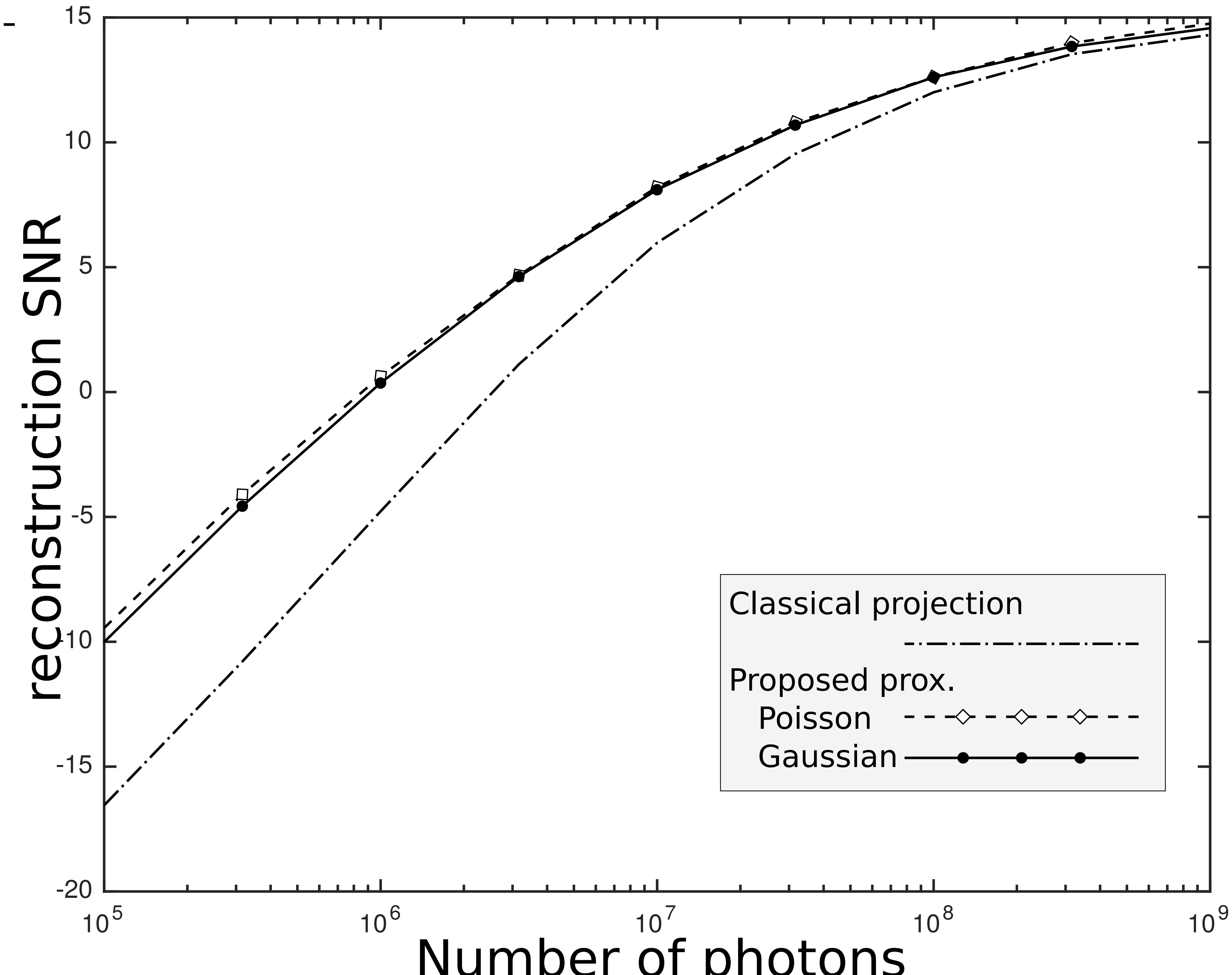
Numerical results : low light condition

Gaussian Vs Poisson model

Dark current b=3e-

Gaussian approximation

$$\sigma^2 = \max(d_k, b)$$



Proximity operator for sum of intensities

Sum of intensities model

- Undersampled fringes, broadband imaging,…

$$d_k = \|\boldsymbol{y}_k\|_2^2 + n_k \,,$$

$$\text{setting } \boldsymbol{y}_k = \eta\,\boldsymbol{u}, \text{ with } \eta \geq 0 \text{ and } \|\boldsymbol{u}\|_2 = 1$$

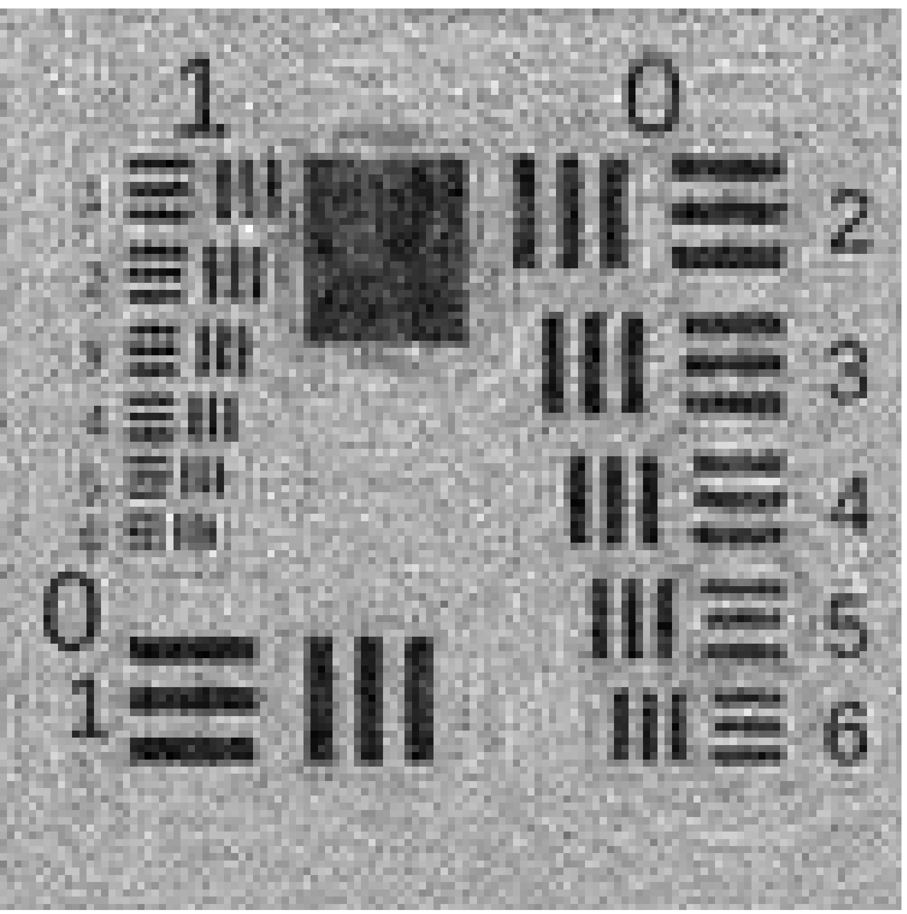
$$\operatorname{prox}_f\left(\widetilde{\boldsymbol{y}}\right) = \begin{cases} \eta^+\,, & \text{if } \|\widetilde{\boldsymbol{y}}\|_2 = 0 \\ \eta^+ \frac{\widetilde{\boldsymbol{y}}}{\|\widetilde{\boldsymbol{y}}\|_2}\,, & \text{otherwise}\,. \end{cases}$$

With
$$\eta^+ = \arg\min_{\eta} \left(\min_{\boldsymbol{u}, \|\boldsymbol{u}\|=1} \left(f\left(\eta\right) + \frac{1}{2}\, \|\eta\,\boldsymbol{u} - \widetilde{\boldsymbol{y}}\|_2^2 \right) \right)\,,$$
$$= \arg\min_{\eta > 0} \left(f\left(\eta\right) + \frac{1}{2}\, \left(\eta - \|\widetilde{\boldsymbol{y}}\|_2\right)^2 \right)\,,$$

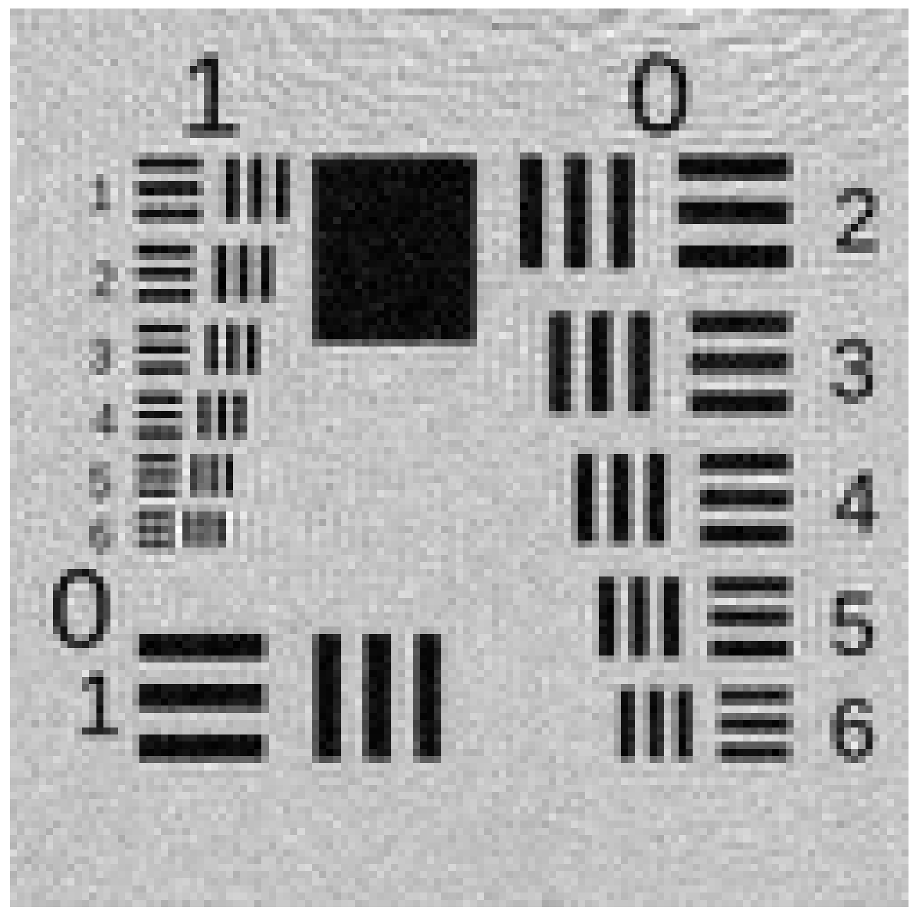
Proximity operator for sum of intensities

- **Results: trading noise for resolution**

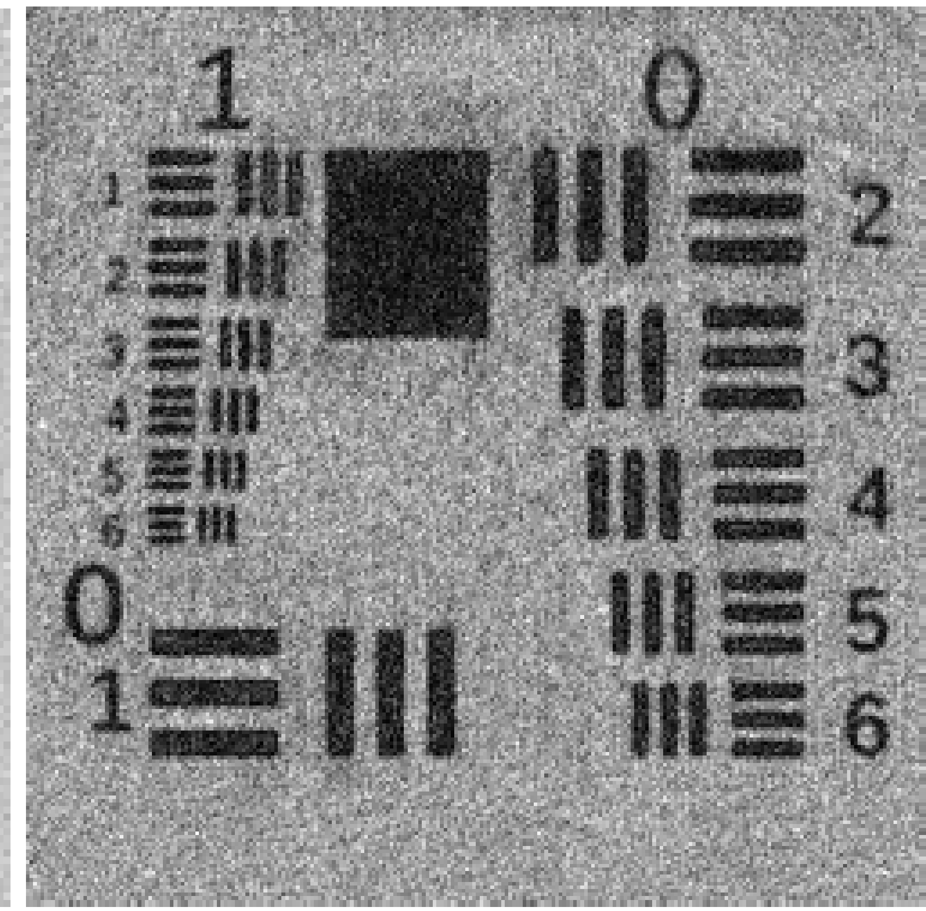
- Data down-sampled by 2x2
- Measurements at 8 different depths



Classical,
2 planes



Classical,
8 planes



Super resolution
8 planes

Image backpropagated at z_0

Telescope tomography

Pros

— Optically motivated mode (restriction to the plausible PSF only),
— Irreducible formulation: should be immune to binarity bias,
— Gives the PSF everywhere (no interpolation needed),
— Can easily take into account variation of aberrations with time.

Cons

— Only accounts for optical aberrations (no jitter, CTI,…),
— Quite computationally intensive (but can be easily parallelized),
— Propagation model needs to be accurate,
— Needs a good knowledge on Euclid optical design,
— Yet to be tested (broadband and under sampled case).