3D Spherical Wavelets for the Study of Wide Galaxy Surveys

Banff CosmoStat2013

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Introduction

A few words about **sparsity**:

- Some data is said to be **sparse** in some **dictionary** when it can be represented by a **small number of coefficients** in this dictionary.
- **Once** the sparse hypothesis is verified you have access to extremely powerful tools to treat your data.
- Example of applications: Denoising, Detection, Component Separation, Linear Problem Inversions, Data Compression...

$\Longrightarrow$ But it all rely on having **appropriate dictionaries** for the data.

Introduction

- In recent years all sky survey have prompted the development of new **multiresolution transforms on the sphere**.
- Building blocks of powerful sparsity based tools but limited to the sphere.
- 3D multiresolution transforms readily expressed in **spherical coordinates** are key to extend the tools on the sphere to the 3D space.

Aim of this work

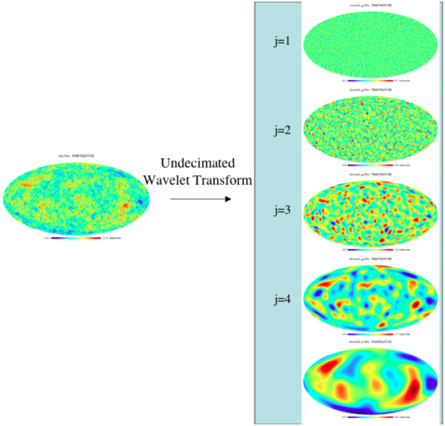
Generalize a wavelet transform defined on the sphere to Spherical 3D using the Spherical Fourier-Bessel framework.

Isotropic Undecimated Wavelet Transform

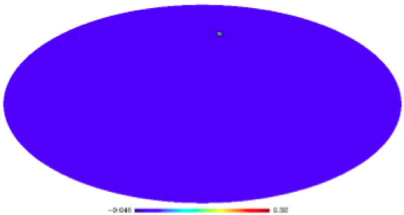
Starck et al. (2006) introduced an **invertible** isotropic wavelet transform on the sphere using **spherical harmonics**.

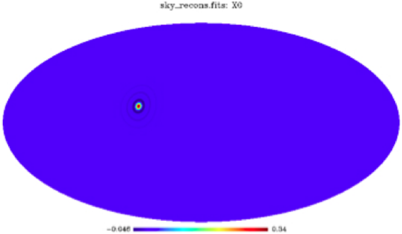
IUWT features

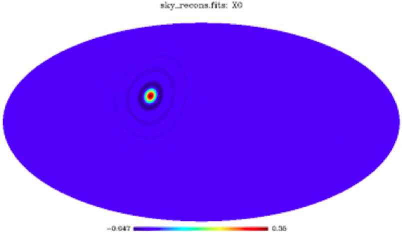
- **Exact reconstruction** formula.
- **Fast** implementation thanks to the HEALpix package.
- **Dictionary adapted to isotropic features**.
- Well behaved in the direct space (limited oscillations).
- Proved to be **very efficient for restoration** purposes.



Isotropic Undecimated Wavelet Transform







Example of application: Poisson denoising on the sphere

Using the IUWT, Poisson denoising of isotropic sources on the Sphere was addressed in Schmidt et al. (2010) for Fermi data.

MS-VSTS

Multi-Scale Variance Stabilization Transform on the Sphere.
Combines a square root variance stabilization transform the IUWT.

Can also be combined with:

- Inpainting
- Source detection
- Multi-Channel Deconvolution

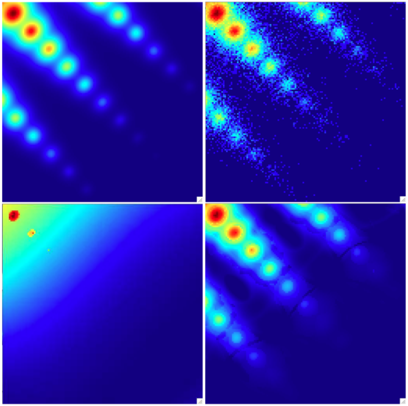
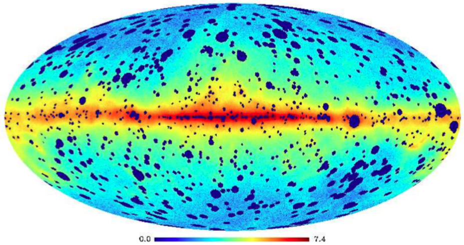
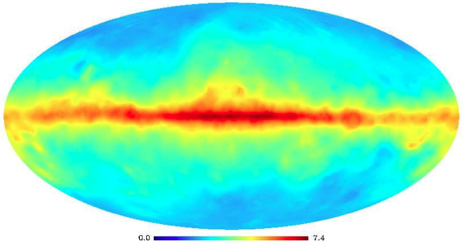


Fig. 5. Comparison of MS-VSTS with Anscombe + wavelet shrinkage on a single face of the first scale of the HEALPix pixelization (angular extent: $\pi/3sr$). *Top Left* : Sources of varying intensity. *Top Right* : Sources of varying intensity with Poisson noise. *Bottom Left* : Poisson sources of varying intensity reconstructed with Anscombe + wavelet shrinkage. *Bottom Right* : Poisson sources of varying intensity reconstructed with MS-VSTS.

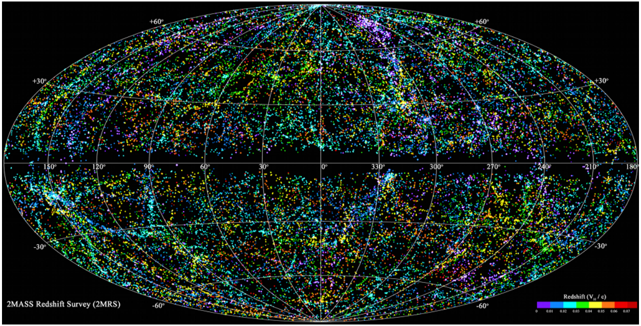
Example of application: Inpainting



Simulated Fermi data denoised and inpainted by MS−VSTS+IUWT



Why the Spherical Fourier-Bessel Transform ?



Interest of the SFB transform

- **Naturally arises** when dealing with 3D data in spherical coordinates.
- Basis functions are **eigenfunctions** of the Laplacian operator.
- **Redshift space distortions** are readily expressible in the SFB basis.
- **Different probes** can be expressed in this basis (BAOs, WL, ISW).

The Spherical Fourier-Bessel Transform

The Spherical Fourier-Bessel Transform of f is its development onto the following orthogonal basis:

$$\Psi_{lmk}(r,\theta,\phi) = \sqrt{\frac{2}{\pi}} j_l(kr) Y_l^m(\theta,\phi) \tag{1}$$

Spherical Fourier-Bessel Transform

Direct Transform:

$$\hat{f}_{lm}(k) = \sqrt{\frac{2}{\pi}} \int_\Omega \int f(r,\theta,\phi) \underbrace{r^2 j_l(kr) dr}_{\text{Spherical Bessel}} \underbrace{\overline{Y}_l^m(\theta,\phi) d\Omega}_{\text{Spherical Harmonics}} \tag{2}$$

Inverse Transform:

$$f(r,\theta,\phi) = \sqrt{\frac{2}{\pi}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int \hat{f}_{lm}(k) k^2 j_l(kr) dk Y_l^m(\theta,\phi) \tag{3}$$

IUWT in the SFB framework

All we need is to be able to express the convolution of f with a scaling function as a function of Spherical Fourier Bessel coefficients.

Isotropic Low-Pass filtering in SFB

Scaling function $\phi^{k_c}(r, \theta_r, \phi_r)$ with cut-off k_c and spherical symmetry:

- $\hat{\phi}^{k_c}_{lm}(k) = 0$ as soon as $(l, m) \neq (0, 0)$
  $\hat{\phi}^{k_c}_{00}(k) = 0$ for all $k \geq k_c$
- $\boxed{\widehat{(f * \phi)}_{lm}(k) = \sqrt{2}\pi \hat{\phi}_{00}(k) \hat{f}_{lm}(k)}$

$\Longrightarrow$ Applying a 3D isotropic low-pass filter is equivalent to **multiplying** the SFB coefficients by a **function of k only**.

IUWT in the SFB framework

- $c^j(r, \theta_r, \phi_r)$ are a sequence of smooth approximations of $f(r, \theta_r, \phi_r)$
- c^j are expressed as the convolution of $f(r, \theta_r, \phi_r)$ with $\phi^{2^{-j}k_c}$:

$$\begin{aligned} c^0 &= \Phi^{k_c} * f \\ c^1 &= \Phi^{2^{-1}k_c} * f \\ &\dots \\ c^j &= \Phi^{2^{-j}k_c} * f \end{aligned} \tag{4}$$

- Based on the "a trou" algorithm, wavelet coefficients are the difference between two successive smoothed approximations:

$$w^{j+1}(r, \theta_r, \phi_r) = c^j(r, \theta, \phi) - c^{j+1}(r, \theta, \phi) \tag{5}$$

IUWT in the SFB framework

Recursive definition of the wavelet decomposition

If $\hat{c}^0_{lm}(k) = \hat{f}_{lm}(k)$ then:

$$\hat{c}^{j+1}_{lm}(k) = \hat{h}^j_{00}(k)\hat{c}^j_{lm}(k) \quad (6)$$

$$\hat{w}^{j+1}_{lm}(k) = \hat{g}^j_{00}(k)\hat{c}^j_{lm}(k) \quad (7)$$

Spherical Fourier-Bessel coefficients of the Wavelet decomposition:

$$\left\{\hat{w}^1, \hat{w}^2, \dots, \hat{w}^{J-1}, \hat{c}^J\right\}$$

with $\boxed{\hat{h}^j_{00}(k) = \frac{\hat{\Phi}^{2^{-(j+1)}k_c}_{00}(k)}{\hat{\Phi}^{2^{-(j)}k_c}_{00}(k)}}$ and $\boxed{\hat{h}^j_{00}(k) = 1 - \frac{\hat{\Phi}^{2^{-(j+1)}k_c}_{00}(k)}{\hat{\Phi}^{2^{-j}k_c}_{00}(k)}}$

IUWT in the SFB framework

Recursive definition of the filtered wavelet reconstruction

Given $\left\{\hat{w}^1, \hat{w}^2, \ldots, \hat{w}^{J-1}, \hat{c}^J\right\}$:

$$\hat{c}^j_{lm}(k) = \hat{c}^{j+1}_{lm}(k)\hat{\tilde{h}}^j_{lm}(k) + \hat{w}^{j+1}_{lm}\hat{\tilde{g}}^j_{lm}(k) \tag{8}$$

which yields $\hat{c}^0_{lm}(k) = \hat{f}_{lm}(k)$.

where $\hat{\tilde{h}}^j$ and $\hat{\tilde{g}}^j$ sont définis par:

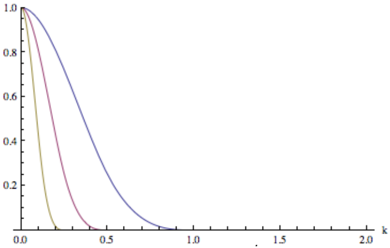
$$\hat{\tilde{h}}^j_{lm}(k) = \frac{\overline{\hat{h}^j}_{lm}(k)}{|\hat{h}^j_{lm}(k)|^2 + |\hat{g}^j_{lm}(k)|^2} \tag{9}$$

$$\hat{\tilde{g}}^j_{lm}(k) = \frac{\overline{\hat{g}^j}_{lm}(k)}{|\hat{h}^j_{lm}(k)|^2 + |\hat{g}^j_{lm}(k)|^2} \tag{10}$$

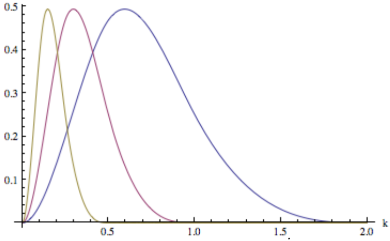
IUWT in the SFB framework

Choice of the scaling function

Any scaling function verifying the spherical symmetry and cutoff frequency will do.
As with the transform on the sphere, we use a 3rd order B-Spline for it's good behavior in direct space.

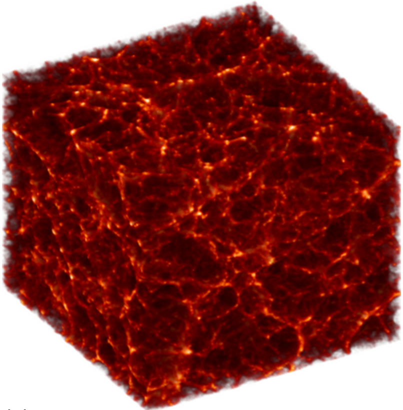


(a) Scaling function $\hat{\Phi}_{00}^{2^{-j}k_c}(k)$ for $j = 0, 1, 2$

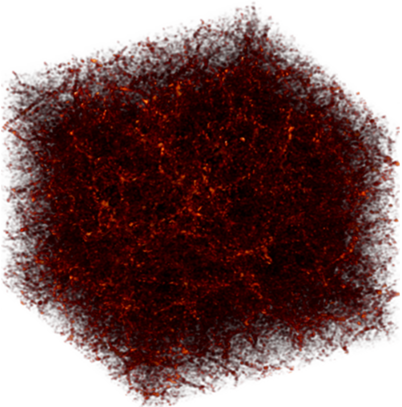


(b) Wavelet function $\hat{\Psi}_{00}^{2^{-j}k_c}(k)$ for $j = 0, 1, 2$

Spherical 3D Isotropic Undecimated Wavelet

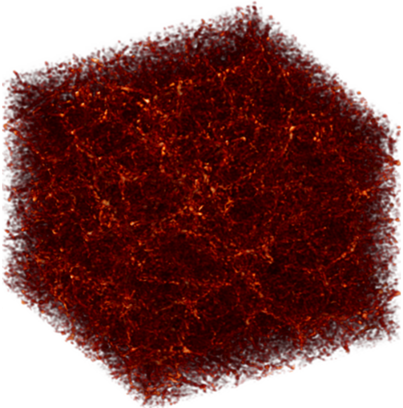


(c) Reconstructed density cube from a_{lmn}

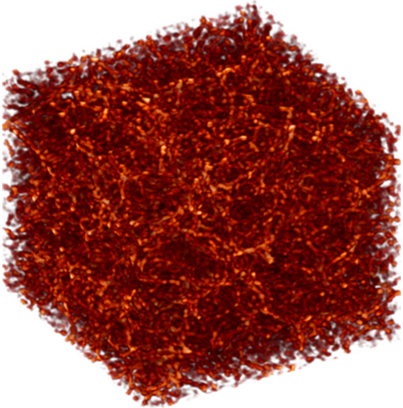


(d) First wavelet scale

Spherical 3D Isotropic Undecimated Wavelet

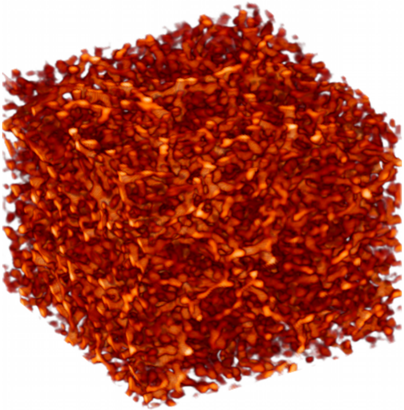


(e) Second wavelet scale

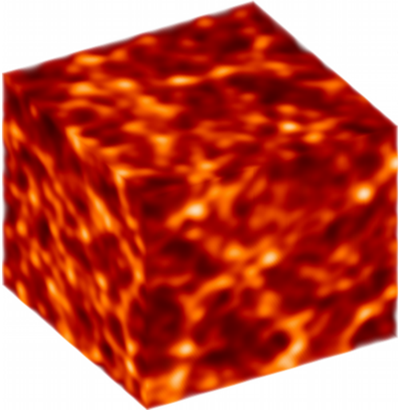


(f) Third wavelet scale

Spherical 3D Isotropic Undecimated Wavelet



(g) Fourth wavelet scale



(h) Smoothed density

Well that's nice but...

This thing is unpractical !

2 main problems:

- This only gives a continuous definition of the wavelet transform.
- A lot of algorithms are iterative and require back and forth wavelet transforms.

$\Longrightarrow$ In practice you need a **discrete sampling scheme** of (r, θ, ϕ) AND (l, m, k) which allows for **back and forth SFB transform**.

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$\Longrightarrow$ In practice you need a **discrete sampling scheme** of (r, θ, ϕ) AND (l, m, k) which allows for **back and forth SFB transform**.

Luckily 2 magical things happen:

- **HEALPix angular discretization scheme** allowing for fast SHT transform in the SFB transform as demonstrated in **Leistedt et al. (2012)**
- When assuming **boundary conditions on the field**, the k **dimension** in the Spherical Bessel Transform can be **discretized**.

The Discrete Spherical Bessel Transform

For the **radial part of the transform**, using **boundary conditions** on $f(r)$ and $\hat{f}(k)$ we showed that, to a good approximation, the Spherical Bessel Transform can be expressed as discrete sums:

$$\hat{f}_{ln} = K^{-3} \sum_{p=1}^{N} f_{lp} \frac{\sqrt{2\pi}}{j_{l+1}^2(q_{lp})} j_l\left(\frac{q_{lp} q_{ln}}{q_{lN}}\right) \quad (11)$$

$$f_{ln} = R^{-3} \sum_{p=1}^{N} \hat{f}_{lp} \frac{\sqrt{2\pi}}{j_{l+1}^2(q_{lp})} j_l\left(\frac{q_{lp} q_{ln}}{q_{lN}}\right) \quad (12)$$

where q_{ln} is the nth zero of the spherical bessel function of order j and $q_{lN} = KR$

$\Longrightarrow$ This defines a matrix transformation between $f(r)$ and $\hat{f}(k)$ sampled on discrete grids $\boxed{r_{ln} = \frac{q_{ln}}{K}}$ and $\boxed{k_{ln} = \frac{q_{ln}}{R}}$

The Discrete Spherical Fourier Bessel Transform

The 2 ingredients of the DSFBT:

- **Angular transform** : HEALpix grid and SHT transform
- **Radial transform** : Discrete Spherical Bessel grid in k and r and transform

$$\begin{bmatrix} \hat{f}_{l1} \\ \hat{f}_{l2} \\ \vdots \\ \hat{f}_{lN} \end{bmatrix} = \frac{1}{K^3} T^l \begin{bmatrix} f_{l1} \\ f_{l2} \\ \vdots \\ f_{lN} \end{bmatrix}$$

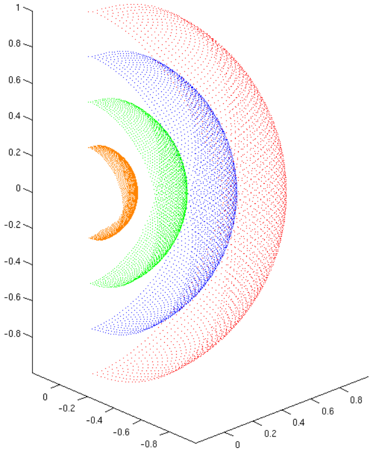
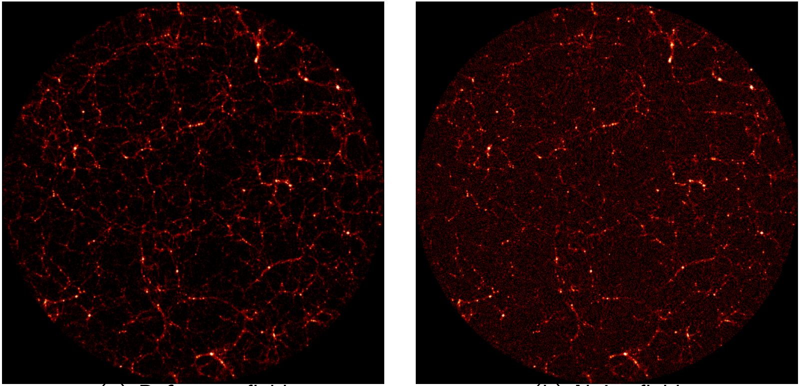


Figure: Spherical 3D grid

$$\boxed{f\left(r_{l_0 n}, \theta_{pix}, \phi_{pix}\right) \Longleftrightarrow \hat{f}\left(l, m, k_{ln}\right)}$$

Toy Experiment: Denoising by hard thresholding

We extracted a density field from an Nbody simulation, added gaussian noise and expanded the field in Spherical Fourier-Bessel Coefficients.



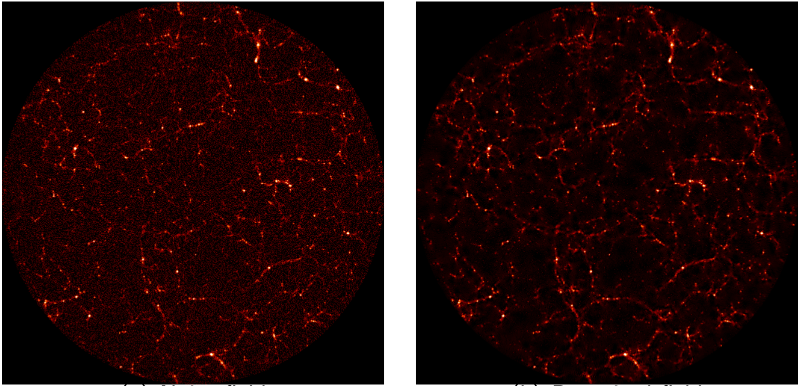
(a) Reference field

(b) Noisy field

Figure: Slice in the reconstruction of the test density field from the original and noisy Spherical Fourier-Bessel coefficients

Toy Experiment: Denoising by hard thresholding

Hard thresholding on the wavelet coefficients is done by setting to zero on each wavelet scale the coefficients below a given $\sigma_T = k\sigma_N$.



(a) Noisy field (b) De-noised field

Figure: Slice in the reconstruction of the noisy and de-noised Spherical Fourier-Bessel coefficients

Toy Experiment: Denoising by hard thresholding

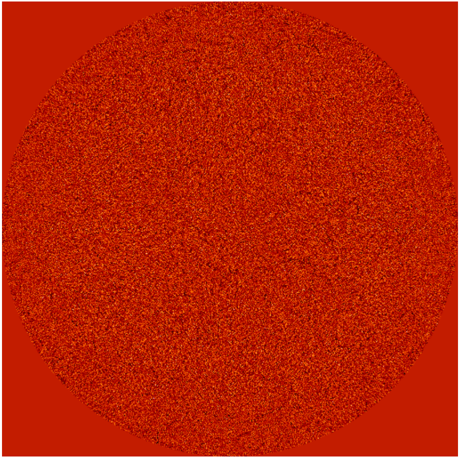


Figure: Difference between noisy and de-noised fields

- (Almost) No discernable features in the residuals.
- The Wavelet Transform has successfully been able to capture the information.

Let's sum up

- The IUWT on the sphere has successfully been extended to the 3D space in spherical coordinates.
- The SFB formalism is very appropriate to the study of wide galaxy surveys.
- This particular wavelet could be very useful when dealing with roughly isotropic feature in the galaxy field.
- A Discrete Spherical Fourier-Bessel Transform has been introduced with a 3D grid compatible with the SFB.
- Other very useful transforms on the sphere could be extended in 3D (Curvelets and Ridgelets) using a similar approach.

All the codes for computing wavelets is contained in a parallelized C++ package with an IDL interface: **http://jstarck.free.fr/mrs3d.html**

3DEX: Fast Spherical Fourier-Bessel decomposition of 3D surveys, Leistedt et al. (2012): **https://github.com/ixkael/3DEX**

One final thought

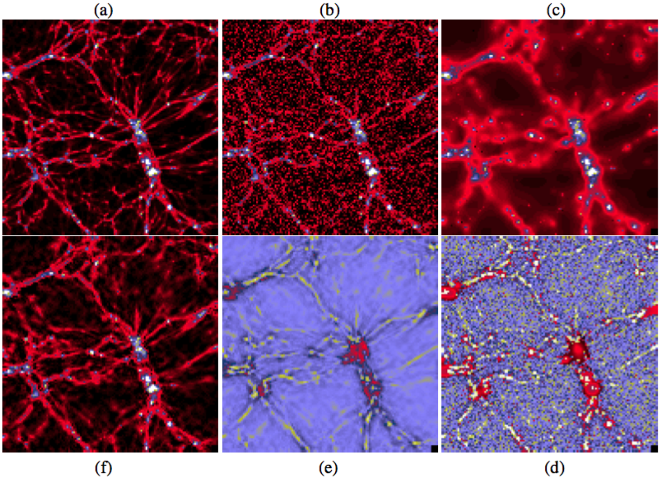
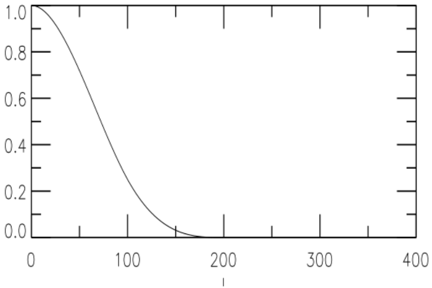


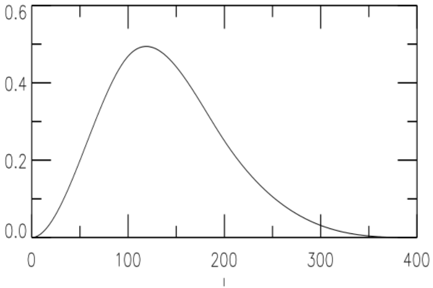
Figure 10: (a) slice of the original λCDM cube, (b) noisy cube, (c) isotropic *à trou* wavelet filtering, (d) residual (b-c), (e) BeamCurvelet filtering of (d), and (f) global filtering (c+e) of (b). (Note : Images (a,b,c,f) are on the same positive color scale – black to white –, while (d,e) are zero – sky blue – mean).

Isotropic Undecimated Wavelet Transform

Transform using the "a trou" algorithm:

- Smoothed approximations are obtained by applying an **axisymetric low pass filter** to the SHT coefficients.
- The Wavelet coefficients are extracted by taking the difference between 2 consecutive smoothed approximations.

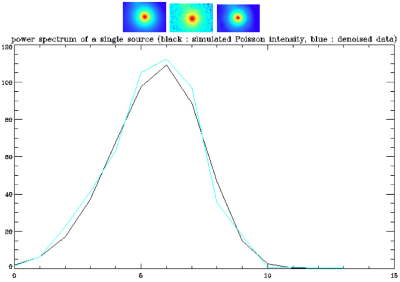




How to extend these wavelets to third dimension ?

2D-1D transform

The idea: Peforming an IUWT in the angular domain followed by a 1D wavelet transform in the 3rd dimension.
Third dimension may be for instance Time or Energy.



This approach is not well suited for 3D spherical fields

For 3D fields, you want to handle the radial dimension in a coherent way with the angular domain.

$\Longrightarrow$ We adopt a true 3D transform using the Spherical Fourier Bessel Transform

Development of Surveys into almns

The redshift survey gives us θ, ϕ, z. We are using Boris' method parallelized on the cluster, computes almns for lmax=512, mmax=512, nmax=512 and Nside=512 in 45min for a survey of 43477 galaxies.

1. We convert the redshifts to comoving distance assuming a given cosmology
2. Assuming an Healpix discretization scheme for (θ, ϕ) we compute in each angular direction the Spherical Bessel Transform

$$f(k_n, \theta, \phi) = \sqrt{\frac{2}{\pi}} \int f(r, \theta, \phi) j_l(k_n r) r^2 dr \tag{13}$$

3. We compute the Spherical Harmonics Transform using Healpix

$$a_{lm}(k_n) = \int f(k_n, \theta, \phi) \overline{Y}_{lm}(\theta, \phi) d\Omega \tag{14}$$

Development of Surveys into almns

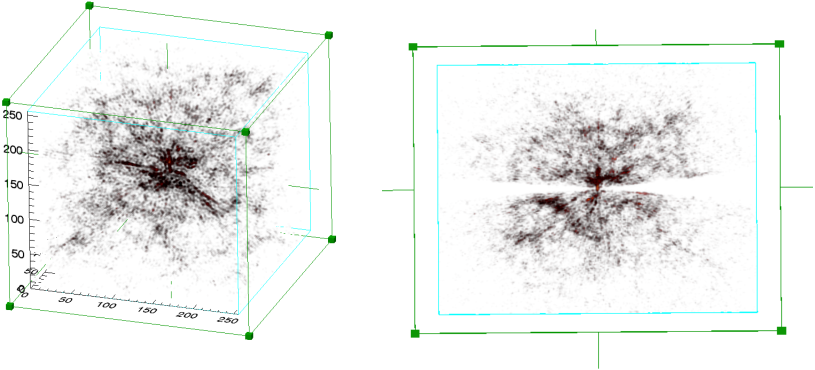


Figure: 2MRS survey reconstrcuted from its almns